The Aggregation Dilemma: How Best to Restructure Sovereign Bonds

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Abstract

Most sovereign bonds today include "aggregated" collective action clauses (CACs). We study the problem of a government that seeks to restructure multiple bond series in the presence of such provisions. We characterize how the optimal aggregation procedure and restructuring offers depend on the heterogeneity of bondholders within and across series, and on the relative size of the bonds. We then analyze how aggregated CACs affect the bond market equilibrium before restructuring when investor bases are determined endogenously. Our results shed light on the aggregation method employed by Argentina and Ecuador in 2020, and on the ongoing reform of Euro-Area CACs.

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Introduction

For over two decades, policy efforts aimed at ensuring the timely and orderly resolution of sovereign debt distress have revolved almost entirely around collective action clauses (CACs).[1] These contractual provisions enable the affirmative vote of a qualified majority of bondholders to bind a dissenting minority to the terms of a restructuring proposal—emulating the majority voting provisions of corporate insolvency laws. CACs have been systematically inserted in the international bonds of major emerging market sovereign issuers with a view to alleviate the "holdout" problem in sovereign debt workouts, whereby a minority of creditors can hinder a restructuring and thus increase the cost of debt distress.[2] Since 2013, the debt securities of Euro-Area governments also incorporate majority amendment provisions.

In practice, the formulation of collective action clauses varies and has evolved over time. The now standard "enhanced" CACs provide that, when attempting to restructure multiple bond series, the sovereign can choose among a menu of three voting procedures (or "modification methods") to determine which series are restructured according to the terms of the proposal (see ICMA (2014)). Specifically:

i. the *series-by-series* voting procedure operates separately within each series, with a supermajority threshold usually set at 75%;

ii. the *"two-limb"* method relies both on the voting outcomes within each series and on the aggregate outcome across series—the voting thresholds in this hybrid procedure being typically set at 50% and 66 2/3%, respectively;

iii. the *"single-limb"* procedure only relies on the aggregate vote across series, with a supermajority threshold of 75% and the additional constraint—known as *uniform applicability* condition—that all bond series receive the same restructuring terms.

In view of their widespread adoption,[3] this paper aims to provide an economic analysis of enhanced CACs. Taking their specification as given, our objective is to elucidate how the sovereign can best take advantage of these voting provisions in a debt workout and to cast light on their broader implications. To do so, we consider an environment with multiple bond series and heterogeneous bondholders—allowing for heterogeneity both within and across series. Whereas the former type of heterogeneity may arise from differences in discount rates,

[1] In 2002, IMF's First Deputy Managing Director Anne Krueger envisioned a statutory bankruptcy regime for countries, but the idea of a "sovereign debt restructuring mechanism" failed to gather political support. More recently, the Common Framework endorsed by the G20 in November 2020 focuses on the coordination among official lenders, as well as between official and private creditors, for low-income countries.

[2] See Eichengreen and Portes (1995) for early proposals towards better coordination among creditors in the resolution of sovereign debt distress. For further background information on sovereign CACs, see among others Buchheit and Gulati (2002) and Gelpern et al. (2016), as well as IMF (2014).

[3] Since their introduction in 2014, enhanced CACs have been inserted in 91% of new international sovereign bonds issued by emerging markets and low-income countries (IMF (2020)).

balance sheets, regulation, information, or litigation skills, the latter can be due to differences across bonds in terms of maturity or coupon rate, as well as in their investor bases. Our contribution is twofold: first, we characterize the choice of aggregation method that minimizes the restructuring cost for the debtor government; second, we analyze how the design of CACs and the anticipation of their optimal use by the government affects the bond market equilibrium prior to restructuring.

In our baseline analytical framework, we consider the problem of a sovereign that seeks to restructure multiple bonds held by atomistic investors. When deciding whether to accept a restructuring offer, bondholders have heterogeneous reservation values. We initially take the distribution of reservation values as given for each bond. Relative to the two-limb procedure, single-limb aggregation comes with a benefit arising from the removal of series-by-series constraints, but also with a cost as it prevents differentiated offers across bonds. In a two-bond case, we characterize how the cost-minimizing aggregation procedure and restructuring offers depend on the voting thresholds, the heterogeneity across bonds, and their relative sizes.[4]

In particular, we show that resorting to single-limb aggregation is valuable when one bond is held by investors who are especially demanding in terms of recovery value and this bond is small in the restructuring pool. Under such conditions, it can be optimal for the government to use the single-limb procedure and make a uniform offer that attracts a low consent share from that bond. However, when the bonds are relatively homogeneous, or when the "expensive" bond—i.e., the bond whose holders tend to have higher reservation values—is large, the series-by-series constraints are not binding, and single-limb voting only comes at a cost due to the uniform applicability restriction. Besides, the single-limb method typically entails a higher aggregate threshold, which contributes to making it less appealing.

This first set of results sheds light on the approach employed by Argentina and Ecuador to restructure their international bonds in August 2020. Then, in the first two instances where enhanced CACs were ever tested, both governments opted in favor of two-limb aggregation.[5] This move took many commentators by surprise. Indeed, in view of its highly effective use in the context of the Greek private restructuring of 2012 (Zettelmeyer et al. (2013)), the presumption in the drafting of enhanced CACs was that the newly-introduced single-limb procedure would become the method of choice to conduct distressed bond exchanges.[6] Instead, our analysis establishes that single-limb aggregation dominates only under very specific circumstances.

[4]For standard threshold values, the series-by-series method is dominated by two-limb voting, hence our analysis focuses on the optimal cross-series aggregation procedure.

[5]For a detailed account of the 2020 Argentine restructuring, in particular, see Buchheit and Gulati (2020).

[6]See, e.g., IMF (2014), Gelpern et al. (2016), and Sobel (2016). The introduction of the single-limb procedure constituted the real innovation in the drafting of enhanced CACs. Series-by-series CACs had been inserted in English-law governed sovereign issues in the 1990s, and then adopted under New York law in the early 2000s. The two-limb method was first introduced in 2003 in the sovereign bonds of Uruguay.

The second part of our analysis embeds the government's restructuring problem into two distinct equilibrium settings to investigate the implications of aggregated CACs on the sovereign bond market in contexts where investor bases and reservation value distributions are determined endogenously. In the first extension of our baseline setup, we analyze how the design of CACs affects the potential entry of (non-atomistic) vulture funds—who may be able to block a restructuring by acquiring sufficiently large positions. We show that, even though it may not be employed in equilibrium, the single-limb procedure can nonetheless play an important role as an *off-equilibrium* deterrent to vultures' holdout attempts.

Taken together, the aforementioned results imply that the two aggregation methods should be viewed as complementary tools in the restructuring arsenal, and shed a critical light on the ongoing reform of Euro-Area sovereign CACs. By virtue of the Treaty establishing the European Stability Mechanism (ESM), all Euro-Area public debt securities issued after January 2013 incorporate two-limb CACs.[7] In November 2021, however, Euro-Area governments agreed, as part of an amendment of the ESM Treaty, to replace two-limb-only with single-limb-only CACs.[8] Our analysis suggests that combining single-limb and two-limb aggregation, as done under the standard enhanced CACs, would make the debtor country strictly better off: whereas the former discourages the entry of vulture funds, the latter often turns out to be the optimal restructuring method once vultures have been kept at bay.

In the second extension of our baseline setup, bonds explicitly differ in terms of their maturities and coupon rates, and the bond market is populated by a continuum of investors with heterogeneous discount rates. Investors sort into bonds in an initial trading stage and, in the event of a restructuring, the government optimally chooses the recovery values and modification method. In equilibrium, the anticipation of the government's restructuring approach affects the outcome of the trading stage, and the government's restructuring approach itself depends on the bond-specific reservation value distributions—determined by the sorting of investors in the prior stage. In this setting, when assessing the impact of the debt maturity composition on the equilibrium restructuring procedure and payouts across bonds, one must take into account how a change in relative sizes affects the heterogeneity in reservation value distributions via the endogenous sorting of investors across maturities. We provide a numerical illustration where an increase in the relative size of the more "expensive" bond actually plays in favor of using the single-limb aggregation procedure, by increasing the degree of heterogeneity in reservation values across bonds. Finally, we illustrate how the design of CACs can affect relative bond valuations ahead of a restructuring in this setup.

[7] Euro CACs have not been tested yet. It is worth noting that, although most Euro-Area debt securities are governed by domestic law, the jurisprudence of European courts in the wake of the Greek restructuring sets important limits to potential governmental interference with bondholder's property rights (Grund (2017)).

[8] The decision will enter into force once the parliaments of all member states have ratified the Amending Agreement. At the time of writing, Italy is the only member state that has not yet ratified the agreement.

Related Literature. Our work contributes to the broad analysis of how the legal environment shapes the sovereign debt market. The seminal paper by Eaton and Gersovitz (1981) puts at center stage the lack of enforceability stemming from the absence of an international bankruptcy court and from the legal doctrine of sovereign immunity.[9] Bolton and Jeanne (2007) analyze how the weak contractual environment and lack of a bankruptcy regime affect the types of debt claims used in international borrowing and lending. Another important feature of the sovereign debt market—the absence of a seniority structure across debt claims—can be traced back to the common use of pari passu and negative pledge clauses in sovereign debt contracts, giving rise to the debt dilution problem analyzed in Bolton and Jeanne (2009), Chatterjee and Eyigungor (2015), and Hatchondo et al. (2016) among others.

A recent strand of the sovereign debt literature—see, e.g., Yue (2010), Benjamin and Wright (2013), Hatchondo et al. (2014), Dvorkin et al. (2021) and Arellano et al. (2023)—focuses on the restructuring process, studying its quantitative importance and examining the determination of the level of haircuts and length of negotiations.[10] In particular, Pitchford and Wright (2012) analyze how the type of settlement process affects delays in an environment where the government cannot commit to settling on worse terms with holdouts. Instead, we zoom in on the role of contractually specified voting procedures in determining payouts in sovereign workouts, and analyze their impact on the bond market ahead of the restructuring.

Existing theoretical work on majority amendment provisions in sovereign bonds—such as, e.g., Bi et al. (2016), Engelen and Lambsdorff (2009), and Haldane et al. (2005)—concentrates on series-by-series CACs. Considering setups featuring one bond and a single set of creditors, these papers study how strategic interactions and restructuring outcomes are affected by the introduction of a supermajority rule in place of a unanimity requirement.[11] Closer to our paper, Bond and Eraslan (2010) analyze the optimal choice of voting rule by the debtor government. By design, this prior work is silent on cross-bond heterogeneity and aggregation. Instead, we adopt a setting with multiple bond series to address issues specifically related to the now widely adopted enhanced CACs and to the design of aggregated Euro CACs.[12]

On the empirical front, while the literature on private debt workouts—see, e.g., Cruces and Trebesch (2013)—mostly focuses on aggregate restructuring outcomes, Sturzenegger and Zettelmeyer (2006, 2008) and Zettelmeyer et al. (2013) document within-deal variation in haircuts for selected episodes. More recently, Asonuma et al. (2023) systematically explore the relationship between haircuts and maturity at the bond level, and Fang et al. (2021)

[9]Although sovereigns are no longer immune from suits in key jurisdictions such as the U.S. and the U.K., they are still effectively immune from attachment attempts by judgment creditors.

[10]See also Bulow and Rogoff (1989) for an early analysis of payouts in sovereign debt restructurings.

[11]Ghosal and Thampanishvong (2013) analyze the impact of a change in voting threshold on interim and ex ante efficiency in the presence of debtor moral hazard and incomplete information.

[12]Early analyses of CACs in a one-bond setting are still relevant in practice in situations where a single bond is being restructured, or when the bonds being restructured only feature old-style series-by-series CACs.

analyze the combined impact of CACs and haircuts on participation rates within restructuring episodes. None of these studies investigates specifically the impact of aggregated CACs on restructuring outcomes. As more evidence on the restructuring of bonds with aggregated CACs accumulates in the future, our analysis can provide guidance for further empirical work, especially with regard to the choice of aggregation procedure.

Our paper is also connected to a series of empirical studies that investigate how the inclusion of CACs affects sovereign bond prices, including the early contribution by Eichengreen and Mody (2004). Among recent work on the topic, Picarelli et al. (2019) and Carletti et al. (2021) analyze the pricing of Euro CACs, while Chung and Papaioannou (2021) document the impact of enhanced CACs in normal times and during distress episodes. To our knowledge, no study has yet explored the differential price impact of various forms of aggregated voting provisions, and how it varies across bonds. Theoretical predictions on these effects must rely on a fine understanding of how aggregated CACs play out in a restructuring. Our analysis could guide such investigations.

Outline. The paper proceeds as follows. Section 1 describes the tradeoff that the government faces in choosing its restructuring approach and formulates general sufficient optimality conditions. Section 2 offers an analytical characterization of the optimal aggregation procedure and restructuring proposal as a function of the model primitives in the two-bond case. Section 3 provides closed-form results and numerical illustrations in a parametric example. Section 4 analyzes how the design of CACs affects the potential entry of vulture funds. Section 5 embeds the government's problem in a setting where bond-specific reservation value distributions arise endogenously from the sorting of heterogeneous investors into bonds. Section 6 concludes. All proofs are in the appendix.

1 The Restructuring Problem

This section lays out a general framework to analyze the government's optimal use of modification provisions in the restructuring of multiple bonds, allowing for creditor heterogeneity both within and across bond series.[13]

Bonds, Restructuring Proposal, and Creditor Heterogeneity. There is a countable set $\mathcal{B}$ of bond series to be restructured, with $|\mathcal{B}| \geq 2$. The relative size of bond series i, expressed as a fraction of the face value of the entire restructuring pool, is given by λ_i, with $\sum_{i \in \mathcal{B}} \lambda_i = 1$. A restructuring proposal $\mathbf{w} = \{w_i\}_{i \in \mathcal{B}}$ made by the government consists of series-specific recovery values w_i per unit of face value. Upon receiving an offer from the government, a bondholder accepts if the proposed recovery value w_i is at least as high as her

[13] In practice, a bond series typically corresponds to a unique ISIN, although it may sometimes include bonds with different ISINs but same payment terms.

own idiosyncratic reservation value.[14] To capture within-series creditor heterogeneity as well as cross-series heterogeneity arising from differences in creditor base and/or bond characteristics (e.g., payment terms), we assume that the reservation values of holders of bond series i are distributed according to the cumulative distribution function F_i, known by the government.[15] The share of holders of series i that give their consent to an offer w is thus equal to $F_i(w)$. For expositional simplicity, we assume that all investors are atomistic and that the CDFs are continuous. Section 4 analyzes an extension where a large investor can take blocking positions.

Modification Methods. The sovereign can choose among the three voting procedures outlined in the introduction to implement a restructuring. We use the subscript 0 to denote the series-by-series procedure, and we use the subscripts 1 and 2 to denote the single-limb and two-limb procedures, respectively. Under series-by-series voting, an entire bond series i is restructured if the consent share within this series is greater than or equal to a given threshold τ_0. According to the two-limb procedure, all bond series in the aggregated pool are restructured if the consent share within each series is greater than or equal to the threshold τ_2^{s} and the consent share over the entire pool is no smaller than $\tau_2^{\mathrm{a}} > \tau_2^{\mathrm{s}}$. Finally, under single-limb voting, the uniform applicability condition requires that the same offer be made to all bond series,[16] and CACs are triggered as long as an aggregate threshold τ_1 is reached.

1.1 Government's Problem

The government seeks to restructure all bonds series $i \in \mathcal{B}$ at minimum cost. The set of constraints that need to be satisfied by the restructuring proposal to achieve this objective depends on the elected modification method.[17] With series-by-series voting, the restructuring offer $\mathbf{w} = \{w_i\}_{i \in \mathcal{B}}$ must be such that

$$F_i(w_i) \geq \tau_0 \quad \text{for all } i \in \mathcal{B}.$$

[14] We take these reservation values as exogenous, abstracting from explicit strategic considerations that may affect the expected value of holding out. There exists little evidence on holdout payoffs, apart from well-publicized cases such as the Argentine settlement following the 2001 default—see Cruces and Samples (2016). Schumacher et al. (2021) provide empirical evidence on the incidence of sovereign debt litigation.

[15] In practice, the government learns about reservation values during preliminary talks with bondholder committees. In Appendix B, we consider an environment where the government faces some uncertainty over the consent shares that a restructuring proposal will attract. We show that the presence of uncertainty does not alter the insights from our analysis.

[16] This condition is meant to provide a safeguard to ensure inter-creditor equity, by avoiding that holders of large bond series dictate terms that are discriminatory against smaller series (see IMF (2014)).

[17] We assume that a unique procedure is applied to the entire pool $\mathcal{B}$. In practice, a government may partition $\mathcal{B}$ and use different modification methods on different subsets of bonds.

Under the two-limb method, the offer $\mathbf{w}$ must satisfy the aggregate constraint

$$\sum_{i \in \mathcal{B}} \lambda_i F_i(w_i) \geq \tau_2^{\mathrm{a}}, \tag{1}$$

along with the individual consent requirements

$$F_i(w_i) \geq \tau_2^{\mathrm{s}} \quad \text{for all } i \in \mathcal{B}. \tag{2}$$

Finally, under single-limb aggregation, the "uniform" offer w must be such that

$$\sum_{i \in \mathcal{B}} \lambda_i F_i(w) \geq \tau_1.$$

In choosing a modification method and the offer w_i made to each bond series $i \in \mathcal{B}$, the government wishes to minimize the total payout

$$C = \boldsymbol{\lambda}.\mathbf{w} = \sum_{i \in \mathcal{B}} \lambda_i w_i.$$

We shall proceed under the realistic assumption that

$$\frac{1}{2} \leq \tau_2^{\mathrm{s}} < \tau_2^{\mathrm{a}} \leq \tau_1 \leq \tau_0 < 1. \tag{3}$$

Under (3), two-limb aggregation dominates (at least weakly) series-by-series voting.[18] We will therefore restrict our attention to the optimal choice of aggregation procedure.

1.2 Optimal Restructuring Offers

A preliminary step towards comparing two-limb vs single-limb aggregation consists in characterizing the optimal restructuring proposal under each procedure. The optimal uniform offer u^* under single-limb voting is such that the average consent share, weighted by the bond face values, is equal to the aggregate threshold τ_1, i.e.,

$$\sum_{\mathcal{B}} \lambda_i F_i(u^*) = \tau_1, \tag{4}$$

and the minimum restructuring cost C_1 under single-limb voting is equal to u^*.

[18] In the ICMA standard published in 2014-15, the thresholds are $\tau_0 = \tau_1 = 3/4$, $\tau_2^{\mathrm{s}} = 1/2$, and $\tau_2^{\mathrm{a}} = 2/3$. It is worth noting that, in the Uruguay 2003 version of CACs, the voting thresholds are such that $\tau_2^{\mathrm{s}} < \tau_0 < \tau_2^{\mathrm{a}}$, so that the optimal choice between series-by-series voting and two-limb aggregation is non-trivial.

Instead, the optimal two-limb offer is the solution to the constrained optimization problem

$$C_2 \equiv \min_{\{w_i\}} \sum_{\mathcal{B}} \lambda_i w_i$$

subject to

$$\sum_{\mathcal{B}} \lambda_i F_i(w_i) = \tau_2^{\mathrm{a}}, \qquad \text{and} \qquad F_i(w_i) \geq \tau_2^{\mathrm{s}}, \quad i \in \mathcal{B}.$$

Since $\tau_2^{\mathrm{a}} > \tau_2^{\mathrm{s}}$, the aggregate constraint (1) is binding and holds as an equality. However, it is a priori unclear which of the individual constraints, if any, may be binding.

Auxiliary Problem. It will be useful in our analysis to consider the auxiliary problem

$$\min_{\{w_i\}} \sum_{\mathcal{B}} \lambda_i w_i \qquad \text{subject to} \qquad \sum_{\mathcal{B}} \lambda_i F_i(w_i) = \tau, \tag{5}$$

for a given generic aggregate threshold τ. The government's cost-minimization problem under single-limb and two-limb voting can be construed by reference to this problem. Under single-limb voting, the aggregate threshold is τ_1 and the offer needs to satisfy the additional "uniform applicability" restriction, which simplifies the problem into (4). Instead, under two-limb voting, the aggregate threshold is τ_2^{a} and the government needs to take into account the additional series-by-series constraints (2). Whenever the solution to the auxiliary problem for $\tau = \tau_2^{\mathrm{a}}$ satisfies the latter constraints, it therefore coincides with the optimal two-limb offer. The Lagrangian for the auxiliary problem is

$$\mathcal{L} = \sum_{\mathcal{B}} \lambda_i w_i + \xi\Big(\tau_2^{\mathrm{a}} - \sum_{\mathcal{B}} \lambda_i F_i(w_i)\Big).$$

Assuming differentiability, the first-order condition is

$$F_i'(w_i) = \xi^{-1}, \qquad \text{for all } i \in \mathcal{B}.$$

Intuitively, because a marginal increase dw in the offer to series i raises the aggregate consent share by $\lambda_i F_i'(w_i)dw$ at a cost of $\lambda_i dw$, interior optimality requires that the "bang for the buck" $F_i'(w_i)$ be equalized across all bonds.

1.3 Optimal Voting Procedure: Key Tradeoff

Comparing across procedures, the unique appeal of single-limb voting comes from the fact that it removes the need to satisfy the series-by-series constraints; however, resorting to this method does entail a cost arising from the uniform applicability restriction—let alone the higher aggregate consent threshold when $\tau_1 > \tau_2^{\mathrm{a}}$. These observations immediately deliver sufficient conditions under which two-limb aggregation is optimal.

Proposition 1. *Two-limb aggregation is (at least weakly) optimal if one of the following conditions holds:*

(i) The optimal single-limb uniform offer, given by the unique solution u^* to (4), is such that $F_i(u^*) \geq \tau_2^{\mathrm{s}}$ for all $i \in \mathcal{B}$;

(ii) The solution $\widehat{\mathbf{w}}$ to the auxiliary problem (5) is such that $F_i(\widehat{w}_i) \geq \tau_2^{\mathrm{s}}$ for all $i \in \mathcal{B}$.

Furthermore, if condition (ii) holds, the optimal two-limb offer coincides with $\widehat{\mathbf{w}}$.

Indeed, the unique advantage of single-limb aggregation is worthless if any of the two conditions holds, hence the two-limb procedure dominates in these configurations. Going beyond these general statements requires making further assumptions on the environment—that is, on the number of bonds and their relative sizes λ_i, on the reservation value distributions F_i, and on the various voting thresholds.

2 Two-Bond Case

We now focus on the case where there are only two bond series outstanding, H and L, with relative weights $\lambda_H = \lambda \in (0,1)$ and $\lambda_L = 1-\lambda$. We denote by $F_i : \mathbb{R}^+ \to [0,1]$ the cumulative distribution function of reservation values for bond $i \in \{H, L\}$, which we assume to be strictly increasing and twice differentiable on $\mathbb{R}^+$ with $F_i(0) = 0$, and we denote by f_i the corresponding density function. We assume that holders of bond H tend to have higher reservation values, so that

$$F_H(w) < F_L(w) \quad \text{for all } w > 0, \tag{6}$$

or equivalently

$$F_L^{-1}(\tau) < F_H^{-1}(\tau) \quad \text{for all } \tau \in (0,1).$$

Because it takes a more generous offer to reach a certain approval rate for bond H than for bond L, we will sometimes loosely refer to bond H as the "expensive" bond.

2.1 Single-Limb Aggregation

Under single-limb voting, the government's restructuring proposal must satisfy the uniform applicability condition, requiring the offer to be the same across series. As per (4), the cost-minimizing offer is given by $u^* = u(\lambda, \tau_1)$, where $u(\lambda, \tau)$ is implicitly defined as the unique solution to the equation

$$\lambda F_H(u) + (1-\lambda)F_L(u) = \tau. \tag{7}$$

The following observations obtain immediately.

Lemma 1. *The consent shares under the optimal uniform offer u^* are such that*

$$F_H(u^*) < \tau_1 < F_L(u^*). \tag{8}$$

Moreover, u^* is strictly increasing in τ_1 and in the relative size λ of the expensive bond, with

$$\lim_{\lambda \downarrow 0} u^* = F_L^{-1}(\tau_1) \qquad \text{and} \qquad \lim_{\lambda \uparrow 1} u^* = F_H^{-1}(\tau_1). \tag{9}$$

Next, we establish conditions under which $F_H(u^*) \geq \tau_2^{\text{s}}$, in which case the optimal single-limb offer u^* satisfies both of the series-by-series constraints imposed under two-limb voting.

Lemma 2. *If $F_L^{-1}(\tau_1) \geq F_H^{-1}(\tau_2^{\text{s}})$, the optimal uniform offer u^* is such that $F_H(u^*) \geq \tau_2^{\text{s}}$ for any $\lambda \in (0,1)$. Otherwise, $F_H(u^*) \geq \tau_2^{\text{s}}$ if $\lambda \geq \lambda_\dagger$, where $\lambda_\dagger$ is such that $u(\lambda_\dagger, \tau_1) = F_H^{-1}(\tau_2^{\text{s}})$.*

Naturally, the optimal uniform offer $u^* = u(\lambda, \tau_1)$ is more likely to satisfy the individual constraints for high values of τ_1 and low values of τ_2^{s}. Besides, since the offer u^* induces an average consent share τ_1, greater homogeneity across bonds makes it more likely that $F_H(u^*)$ is close to τ_1 and thus greater than τ_2^{s}. Finally, as captured by the second part of the lemma, a higher relative size λ of the expensive bond H makes it more likely that the constraint on this bond be satisfied by increasing the amount of the uniform offer u^*, with the cutoff value $\lambda_\dagger$ increasing in the degree of cross-series heterogeneity.

2.2 Two-Limb Aggregation

Under two-limb aggregation, the optimal offer is the solution to

$$\min_{w_H,\, w_L} \ \lambda w_H + (1-\lambda) w_L$$

subject to

$$\lambda F_H(w_H) + (1-\lambda)F_L(w_L) = \tau_2^{\text{a}}, \tag{10}$$

$$F_i(w_i) \geq \tau_2^{\text{s}}, \qquad i = H, L. \tag{11}$$

Let us first focus on the aggregate constraint (10), and let $\mathcal{W}$ denote the set of offers w_L to the holders of bond L such that this constraint can be met for some offer w_H to bond H.[19]

[19]When the relative size λ of bond H is sufficiently small, meeting the aggregate consent condition (10) is impossible if w_L is too small (resp., too high), even with an arbitrary large w_H (resp., even when setting

Given an offer $w_L \in \mathcal{W}$ to bond L, the offer w_H that the sovereign needs to make to holders of bond H in order to satisfy the aggregate condition (10) is given by

$$w_H = F_H^{-1}\left(\frac{\tau_2^{\mathrm{a}} - (1-\lambda)F_L(w_L)}{\lambda}\right) \equiv g(w_L). \tag{12}$$

The function g is strictly decreasing in w_L, reflecting the fact that if a more generous offer is made to bond L—thus increasing the consent share for this series—a less appealing offer can be made to the other bond. Furthermore, one can derive the following equivalence relationships from the aggregate condition (10), relating the relative consent shares (and offers) across the two bonds and their absolute levels.

Remark 1. For any pair of offers satisfying the aggregate requirement (10), i.e., for any (w_H, w_L) such that $w_L \in \mathcal{W}$ and $w_H = g(w_L)$,

$$F_H(w_H) < F_L(w_L) \quad \Leftrightarrow \quad F_L(w_L) > \tau_2^{\mathrm{a}} \quad \Leftrightarrow \quad F_H(w_H) < \tau_2^{\mathrm{a}},$$

and

$$w_H > w_L \qquad \Leftrightarrow \qquad w_L < u(\lambda, \tau_2^{\mathrm{a}}) \quad \Leftrightarrow \quad w_H > u(\lambda, \tau_2^{\mathrm{a}}).$$

Auxiliary Problem. To characterize the optimal offer under two-limb aggregation, we consider the corresponding auxiliary problem

$$\min_{w_H,\, w_L} \ \lambda w_H + (1-\lambda) w_L \quad \text{subject to} \quad \lambda F_H(w_H) + (1-\lambda)F_L(w_L) = \tau_2^{\mathrm{a}},$$

and we denote its solution by $\widehat{\mathbf{w}} = (\widehat{w}_H, \widehat{w}_L)$. This problem can be restated more concisely as

$$\min_{w_L \in \mathcal{W}} \ \lambda g(w_L) + (1-\lambda) w_L, \tag{13}$$

and one can show that the objective function in (13) is (strictly) convex when

$$\frac{d \log f_L(w_L)}{d w_L} < \frac{d \log f_H\big(g(w_L)\big)}{d w_L}. \tag{14}$$

In particular, (14) is satisfied if the two densities f_H and f_L are strictly decreasing. Assuming convexity, an interior solution to (13) is pinned down by the first-order condition

$$f_L(w_L) = f_H\big(g(w_L)\big). \tag{15}$$

$w_H = 0$). See Remark A-1 in the appendix for an explicit definition of $\mathcal{W} \equiv \mathcal{W}(\lambda, \tau_2^{\mathrm{a}})$.

Optimal Two-Limb Offer. The optimal two-limb offer coincides with the auxiliary solution $\widehat{\mathbf{w}} = (\widehat{w}_H, \widehat{w}_L)$—such that $\widehat{w}_L$ minimizes (13) and $\widehat{w}_H = g(\widehat{w}_L)$—as long as the latter satisfies the two individual constraints (11), i.e., when

$$\tau_2^{\mathrm{s}} \leq F_L(\widehat{w}_L) \leq \frac{\tau_2^{\mathrm{a}} - \lambda\tau_2^{\mathrm{s}}}{1-\lambda}. \tag{16}$$

The second inequality in (16) guarantees that $F_H(\widehat{w}_H) \geq \tau_2^{\mathrm{s}}$, and is trivially satisfied if $\lambda \geq (1-\tau_2^{\mathrm{a}})/(1-\tau_2^{\mathrm{s}})$. On the other hand, the inequality $F_L(\widehat{w}_L) \geq \tau_2^{\mathrm{s}}$ is necessarily satisfied when λ is sufficiently small. Intuitively, when a bond series is large relative to the total face value of the restructuring pool, the need to reach the high aggregate consent level τ_2^{a} makes it likely that the series-by-series requirement is met for this bond.

Lemma 3. *Suppose that the auxiliary problem (13) is strictly convex with an interior solution. The solution $\widehat{\mathbf{w}}$ to this problem satisfies the series-by-series constraint on bond H when $\lambda \geq (1-\tau_2^{\mathrm{a}})/(1-\tau_2^{\mathrm{s}})$ or if*

$$f_L\left(F_L^{-1}\left(\frac{\tau_2^{\mathrm{a}} - \lambda\tau_2^{\mathrm{s}}}{1-\lambda}\right)\right) \leq f_H\left(F_H^{-1}(\tau_2^{\mathrm{s}})\right), \tag{17}$$

and it satisfies the series-by-series constraint on bond L when $\lambda \leq (\tau_2^{\mathrm{a}} - \tau_2^{\mathrm{s}})/(1-\tau_2^{\mathrm{s}})$ or if

$$f_L\left(F_L^{-1}(\tau_2^{\mathrm{s}})\right) \geq f_H\left(F_H^{-1}\left(\frac{\tau_2^{\mathrm{a}} - (1-\lambda)\tau_2^{\mathrm{s}}}{\lambda}\right)\right). \tag{18}$$

Conditions (17)-(18) ensure that the first-order optimality condition (15) can be met without violating the individual constraints (11). When instead $\widehat{w}_i < F_i^{-1}(\tau_2^{\mathrm{s}})$, the individual constraint on bond i is binding: the optimal two-limb offer sets $w_i = F_i^{-1}(\tau_2^{\mathrm{s}})$ and adjusts the recovery value on the other bond downwards to satisfy the aggregate consent condition.

2.3 Optimal Restructuring Procedure

In view of Proposition 1, we therefore obtain the following result.

Proposition 2. *If either the conditions stated in Lemma 2 or those provided in Lemma 3 hold, two-limb aggregation is optimal.*

Indeed, the conditions stated in Lemmas 2 and 3 guarantee that the optimal uniform offer and the auxiliary solution satisfy all series-by-series constraints, respectively—the unique advantage of the single-limb procedure being worthless in either case. In view of our discussion of Lemma 2, two-limb voting is more likely to dominate for high values of τ_1 and low values of τ_2^{s}, for low levels of heterogeneity across bonds, and for a high relative size λ of the expensive bond.[20] To complement Proposition 2, it is worth noting that

$$\tau_1 > \tau_2^{\mathrm{a}} \qquad \Rightarrow \qquad \lim_{\lambda \downarrow 0} C_1 = F_L^{-1}(\tau_1) > F_L^{-1}(\tau_2^{\mathrm{a}}) = \lim_{\lambda \downarrow 0} C_2, \tag{19}$$

hence when $\tau_1 > \tau_2^{\mathrm{a}}$, two-limb voting also dominates for λ sufficiently small.

The last result of this section establishes sufficient conditions under which single-limb voting dominates.

Proposition 3. *Suppose that* $\tau_1 < F_L\left(F_H^{-1}(\tau_2^{\mathrm{s}})\right)$ *and the two densities* f_H *and* f_L *are strictly decreasing and intersect only once, at a point* $\tilde{w} \in \left(F_L^{-1}(\tau_1), F_H^{-1}(\tau_2^{\mathrm{s}})\right)$*. Then single-limb voting is optimal (i) for* λ *sufficiently small when* $\tau_2^{\mathrm{a}} = \tau_1$*, and (ii) in the neighborhood of the point* $\tilde{\lambda} > 0$ *such that* $u(\tilde{\lambda}, \tau_1) = \tilde{w}$ *when* $\tau_1 - \tau_2^{\mathrm{a}}$ *is not too large.*

The heterogeneity across bonds must be sufficiently large for the premise of Proposition 3 to be satisfied, as will be illustrated in the next section. Contrasting Part (i) of the proposition with (19) reveals that, for low values of the relative weight λ, the optimal voting procedure depends crucially (and in an intuitive way) on whether $\tau_1 > \tau_2^{\mathrm{a}}$ or $\tau_1 = \tau_2^{\mathrm{a}}$. The second part of the proposition points to a special parameter configuration in which the uniform applicability restriction comes at little cost while the unique advantage of single-limb voting is valuable—in which case this modification method is clearly optimal.

3 Parametric Example

We now consider a particular incarnation of the two-bond case where reservation values for each bond are exponentially distributed, that is,

$$F_i(w) = 1 - e^{-\frac{w}{\phi_i}}, \qquad \text{for } w \geq 0, \tag{20}$$

implying that $F_i^{-1}(\tau) = -\phi_i \log(1-\tau)$, for all $\tau \in (0,1)$. As before, we suppose that holders of bond H tend to have higher holdout values. We thus proceed under the assumption that

$$\gamma \equiv \frac{\phi_H}{\phi_L} > 1.$$

[20] In the parametric example of Section 3, we show that the auxiliary solution is also more likely to satisfy the series-by-series constraints for low levels of heterogeneity and high values of λ.

3.1 Closed-Form Analytical Results

Under the exponential specification, the auxiliary problem is strictly convex and the consent shares under the auxiliary solution are given by[21]

$$F_L(\widehat{w}_L) = \frac{\lambda(\gamma-1)+\tau_2^{\mathrm{a}}}{\lambda(\gamma-1)+1} > \tau_2^{\mathrm{a}}, \tag{21}$$

and

$$F_H(\widehat{w}_H) = \frac{\tau_2^{\mathrm{a}} - (1-\lambda)F_L(\widehat{w}_L)}{\lambda} = \frac{1+\lambda(\gamma-1)-\gamma(1-\tau_2^{\mathrm{a}})}{1+\lambda(\gamma-1)} < \tau_2^{\mathrm{a}}. \tag{22}$$

From (21)-(22), one can see that only the individual constraint on bond H may ever be binding in this example. Moreover, the auxiliary consent shares (and corresponding offers $\widehat{w}_L$ and $\widehat{w}_H$) are increasing in the relative size λ of the more demanding bond, and one can check that $F_L(\widehat{w}_L)$ is increasing in γ while $F_H(\widehat{w}_H)$ is decreasing in γ—i.e., the spread in consent shares under the auxiliary solution is increasing in the degree of bond heterogeneity.

Optimal Two-Limb Offer. The optimal offers under two-limb aggregation coincide with the auxiliary solution as long as $F_H(\widehat{w}_H) \geq \tau_2^{\mathrm{s}}$. Since $F_H(\widehat{w}_H)$ is increasing in λ and decreasing in γ, the inequality is more likely to hold for high values of λ and low values of γ. Indeed, one can show that $F_H(\widehat{w}_H) \geq \tau_2^{\mathrm{s}}$

- if $\gamma \leq (1-\tau_2^{\mathrm{s}})/(1-\tau_2^{\mathrm{a}}) \equiv \overline{\gamma}_X$, for all values of λ;
- if $\lambda \geq (1-\tau_2^{\mathrm{a}})/(1-\tau_2^{\mathrm{s}}) \equiv \underline{\lambda} \in (0,1)$, for all values of γ;
- in the remainder of the parameter space for γ sufficiently small or λ sufficiently large.[22]

Conversely, the consent requirement on the expensive bond H is binding when there is sufficient heterogeneity across the two bonds ($\gamma > \overline{\gamma}_X$) and the relative size of bond H is small. Analytical expressions for the optimal two-limb offers in this case are provided in the appendix.

Optimal Voting Procedure. When parameter values are such that $F_H(\widehat{w}_H) \geq \tau_2^{\mathrm{s}}$, two-limb aggregation is optimal since the series-by-series constraints have no bite. The other sufficient condition for two-limb optimality is that the optimal uniform offer u^* satisfies the series-by-series constraints (i.e., $F_H(u^*) \geq \tau_2^{\mathrm{s}}$), in which case the unique advantage of single-limb aggregation is worthless. In the appendix, relying on Lemma 2, we show that this condi-

[21] The analytical derivations for this section are provided in Appendix A.3.

[22] An explicit condition of the form $\lambda \geq \ell_X(\gamma)$ for $\gamma > \overline{\gamma}_X$ is provided in the appendix, see Remark A-3.

tion holds if the heterogeneity parameter γ is below some threshold value $\overline{\gamma}_U$, or alternatively if λ is sufficiently large.[23]

Exploiting Proposition 3 allows us to establish instead a sufficient condition for single-limb optimality. One can show that the premise of the proposition is satisfied for sufficiently high values of γ. Then, single-limb voting is optimal for λ sufficiently close to zero when $\tau_1 = \tau_2^{\mathrm{a}}$. If instead $\tau_1 > \tau_2^{\mathrm{a}}$, as long as $\tau_1 - \tau_2^{\mathrm{a}}$ is not too large, single-limb aggregation is optimal when the relative size λ of bond H is close to the value $\tilde{\lambda}$—given by (A.21) in the appendix—such that the auxiliary solution would require giving the same recovery value to the two bonds, with a consent share for bond H below τ_2^{s}.

When none of the sufficient conditions holds, one can still compare the restructuring costs across the two aggregation procedures. Under two-limb voting, the constraint on bond H must then be binding, and the restructuring cost is given by

$$C_2 = \lambda \phi_H \log\left(\frac{1}{\zeta_2^s}\right) + (1-\lambda)\phi_L \log\left(\frac{1-\lambda}{\zeta_2^{\mathrm{a}} - \lambda\zeta_2^{\mathrm{s}}}\right),$$

where $\zeta_2^j \equiv 1 - \tau_2^j$ for $j \in \{\mathrm{s}, \mathrm{a}\}$. Noting that the optimal single-limb offer u^* is such that

$$F(u^*) \equiv \lambda F_H(u^*) + (1-\lambda)F_L(u^*) = 1 - \lambda e^{-u^*/\phi_H} - (1-\lambda)e^{-u^*/\phi_L} = \tau_1,$$

one can see that single-limb dominates if and only if $F(C_2) > \tau_1$. We thus obtain a necessary and sufficient condition for single-limb optimality in terms of parameter values, namely:

$$\lambda \left(\zeta_2^{\mathrm{s}}\right)^{\lambda} \left(\frac{\zeta_2^{\mathrm{a}} - \lambda\zeta_2^{\mathrm{s}}}{1-\lambda}\right)^{\frac{1-\lambda}{\gamma}} + (1-\lambda)\left(\zeta_2^{\mathrm{s}}\right)^{\lambda\gamma} \left(\frac{\zeta_2^{\mathrm{a}} - \lambda\zeta_2^{\mathrm{s}}}{1-\lambda}\right)^{1-\lambda} < 1 - \tau_1. \tag{23}$$

3.2 Numerical Illustration

For the sake of illustration, we first assume that $\tau_1 = \tau_2^{\mathrm{a}} = 2/3$ and $\tau_2^{\mathrm{s}} = 1/2$, and we set the distributional parameters to $\phi_H = 0.7$ and $\phi_L = 0.2$ (i.e., the average reservation values for the two bonds are 70 and 20 cents on the dollar, respectively). Figure 1 depicts optimal offers (left panel) and consent shares (right panel) under the two aggregation methods as a function of the relative size λ of the expensive bond. Optimal offers and consent shares under two-limb aggregation are represented as dash-dotted blue and dashed red lines for the cheap and expensive bond, respectively, whereas the optimal offer and consent shares under single-limb aggregation are represented by black lines. The shaded colored lines represent the offer and consent shares under the auxiliary solution when it is not feasible. Consider the two-limb modification method first. When λ is large enough, the auxiliary solution is feasible, both exchange offers are increasing in λ and consent shares are larger than the series-by-series

[23]The appendix gives an explicit condition of the form $\lambda \geq \ell_U(\gamma) \in (0,1)$ for $\gamma > \overline{\gamma}_U$.

threshold. When instead λ is low, the series-by-series constraint for the expensive bond binds: w_H is flat, the consent share for bond H equals τ_2^{s}, and the government sets $w_L < \widehat{w}_L$, so as to satisfy the aggregate consent requirement. Under single-limb voting, the optimal uniform offer and the associated consent shares are smoothly increasing functions of λ.

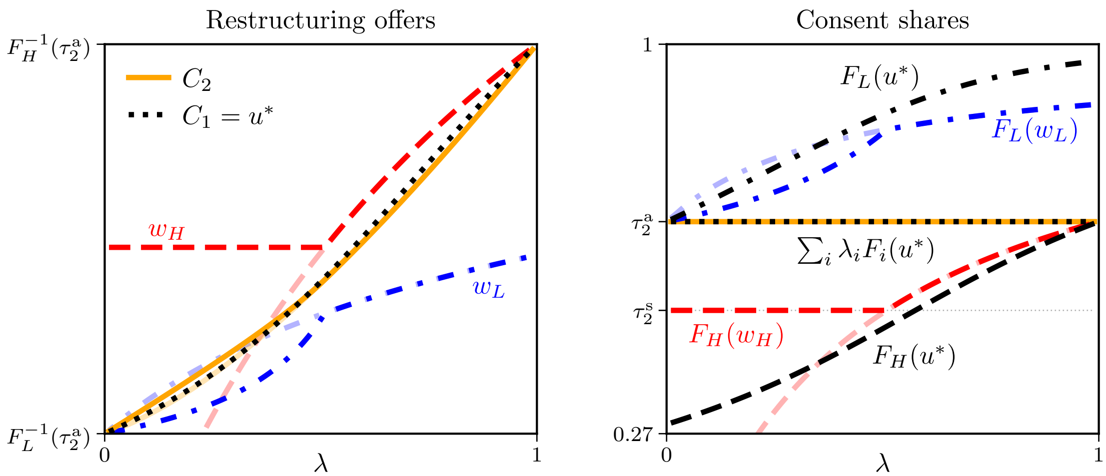

Figure 1: Optimal Restructuring Offers and Consent Shares.

The left panel of Figure 1 also displays the total restructuring cost as a function of λ for each of the two modification methods: single-limb aggregation is represented by the dotted black line (coinciding with u^*), and the cost under the two-limb procedure is depicted by the solid orange line. The figure shows that the two-limb method dominates for sufficiently high values of λ, while single-limb aggregation dominates when the share of the expensive bond is low, consistent with the rest of our analysis.

Finally, Figure 2 represents regions of the parameter space in which either of the two aggregation methods dominates, when $\tau_1 = \tau_2^{\mathrm{a}} = 2/3$ (left panel) and when the thresholds are set as in the standard ICMA CACs, with $\tau_1 = 3/4$ and $\tau_2^{\mathrm{a}} = 2/3$ (right panel). The optimal restructuring method is characterized as a function of the relative size λ of the expensive bond and the degree of heterogeneity across bonds, captured on the y-axis by $\log(\gamma) > 0$. The figure depicts as (colored) dashed, dotted, or dash-dotted lines various analytical objects introduced above, related to sufficient conditions for two-limb or single-limb optimality. The solid (black) boundary is based instead on the necessary and sufficient condition (23) and illustrates the fact that the single-limb procedure is optimal in circumstances when the heterogeneity across bonds is substantial and the relative size of the expensive bond is not too large.[24] Naturally, the region where single-limb voting dominates shrinks when τ_1 increases. The figure also illustrates the fact that when $\tau_1 > \tau_2^{\mathrm{a}}$, two-limb voting always dominates in the neighborhood of $\lambda = 0$, in line with the general observation formulated in (19).

[24]Figure B.4 in Appendix B generalizes this result to an environment with stochastic consent shares.

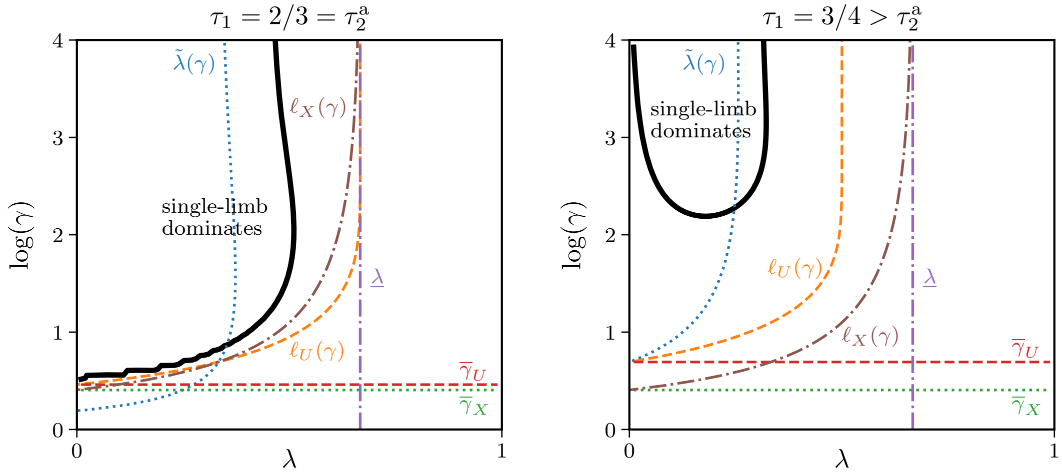

Figure 2: Optimal Voting Rule as a function of Relative Size (λ) and Bond Heterogeneity (γ).

4 CACs and Vultures

This section builds on the two-bond framework to analyze how the design of CACs and their optimal use by the government affect the potential entry of vulture funds who, by taking large positions, may be able to block bond restructurings.

4.1 Setup

We consider an extension of the setup of Section 2 in which, prior to the restructuring episode, a vulture fund may potentially acquire bonds H and L from the continua of heterogeneous investors who hold them. We denote by μ_i the share of bond i acquired by the fund, and we denote by q_i the price paid by the fund per unit of face value. We further assume that the fund needs to pay a fixed transaction cost ε_i to enter the market for bond i, which can be interpreted as the search cost of finding a counterparty.

Blocking Positions. In a restructuring, holding a sufficiently large position effectively allows the fund to prevent the triggering of CACs. We assume that the fund systematically opposes a restructuring when it is able to do so, and that it derives h_i per unit of face value held from blocking the restructuring of bond i.[25] In particular, single-limb aggregation can be blocked if $\sum_i \lambda_i \mu_i > 1 - \tau_1$, and the fund can block the restructuring of bond i under the two-limb procedure when $\mu_i > 1 - \tau_2^{\mathrm{s}}$.

Bond Market Prices. The price q_i at which the fund may acquire a position in bond i naturally depends on the payoff that atomistic investors expect at the subsequent restructuring

[25] We assume that if CACs are not triggered for bond i, then the bond is left unrestructured. That is, CACs thresholds coincide with minimum participation thresholds.

stage. If the market understands that the fund will be unable to prevent the restructuring of bond i, the price q_i coincides with the government's optimal restructuring offer w_i. If instead the fund's acquired positions later enable it to block the restructuring of bond i, investors would ultimately be left with their own reservation values for this bond. In that case, each investor is thus willing to sell as long as the market price q_i is above her reservation value, implying that the fund can acquire a share μ_i of bond i at a cost of $F_i^{-1}(\mu_i)$ per unit of face value.

Cost of Holdout Strategies. The minimum cost at which the fund can acquire a blocking position in the two bonds under either aggregation method is given by[26]

$$\kappa_2 \equiv \inf_{\{\mu_i\}} \sum_i \mathbf{1}_{\{\mu_i>0\}}\Big(\lambda_i \mu_i F_i^{-1}(\mu_i) + \varepsilon_i\Big) \quad \text{s.t.} \quad \sum_i \lambda_i \mu_i > 1 - \tau_1.$$

Likewise, the minimum cost at which the fund may be able to prevent the restructuring of bond i via two-limb aggregation is

$$\kappa_i \equiv \lambda_i(1-\tau_2^{\mathrm{s}})F_i^{-1}(1-\tau_2^{\mathrm{s}}) + \varepsilon_i, \qquad i = H, L.$$

We assume, without loss of generality, that bond L is the cheapest to block, i.e., $\kappa_L < \kappa_H$.[27]

Limited Financial Resources. We assume that the fund has limited resources relative to the size of the outstanding bond series, which restricts its potential ambitions as a holdout. Namely, we assume that the fund's resources e are such that

$$\kappa_L < e < \min\{\kappa_H, \kappa_2\}.$$

We thus focus on the case where the only holdout strategy that the fund may consider is to acquire a blocking position in one bond (namely, bond L) that would prevent the activation of CACs for this bond under two-limb aggregation.

4.2 Equilibrium

In equilibrium, the outcome of the early stage in which the fund decides on (μ_H, μ_L) must be consistent with the outcome of the restructuring stage. To start with, it is straightforward to see that, under the previously stated assumptions, the fund would set $\mu_H = 0$, i.e., it would refrain entirely from entering the market for bond H. Indeed, since any position that the fund may feasibly acquire in that bond would be non-blocking, the price q_H would be equal

[26] Because $\tau_2^{\mathrm{a}} \leq \tau_1$, the cost of blocking both bonds under two-limb aggregation is (at least weakly) larger than under single limb.

[27] Under the maintained assumption (6), we know that $F_L^{-1}(1-\tau_2^{\mathrm{s}}) < F_H^{-1}(1-\tau_2^{\mathrm{s}})$. However, the blocking costs also depend on the size of the bonds (λ_i) and entry costs (ε_i).

to the restructuring payout w_H, and the fixed entry cost would thus be enough to keep the fund at bay. By the same logic, one can see that any position $\mu_L > 0$ that is non-blocking is suboptimal. The only remaining question, therefore, is: would the fund decide to acquire a blocking position in bond L or stay away altogether?

Two-Limb-Only CACs. It is useful to first consider the case where CACs only allow for two-limb aggregation. If the fund contemplates acquiring a blocking position in bond L, the optimal blocking share μ_L^* is given by the solution to the constrained problem

$$B \equiv \sup_{\mu_L > 1-\tau_2^{\text{s}}} \lambda_L \mu_L \Big(h_L - F_L^{-1}(\mu_L) \Big) - \varepsilon_L \quad \text{s.t.} \quad \lambda_L \mu_L F_L^{-1}(\mu_L) + \varepsilon_L \leq e. \tag{24}$$

As long as $B > 0$, it is optimal for the fund to purchase a share μ_L^* of bond L and then block the restructuring of the bond in order to get the holdout payoff h_L, with net benefit B.

Enhanced CACs. Suppose now that the government can choose between the two aggregation methods. How would it respond if the fund builds a blocking position in bond L? Under the two-limb method, bond L would have to be left unrestructured. Instead, under single limb, the government would pick the uniform offer u^* that attracts an aggregate consent share τ_1. As long as the cost of leaving one bond unrestructured is sufficiently high,[28] the government would optimally respond using single-limb aggregation, thus defeating any attempt to block bond L. Hence, the fund would optimally refrain from entering, and the government would choose its restructuring approach according to the logic outlined in Section 2.

4.3 Vulture Entry and the Design of CACs

The analysis of this section reveals that, although single-limb aggregation may not be used in equilibrium under enhanced CACs, it does serve as an *off-equilibrium* threat that deters potential holdouts from acquiring a blocking position in a bond. Indeed, in the equilibrium under enhanced CACs described above, the vulture fund optimally decides to stay at bay. Off equilibrium, the fund considers building a blocking position $\mu_L > 1 - \tau_2^{\text{s}}$ at a cost $q_L = u^*$ per unit of face value, where the market price q_L reflects the (correct) expectation that the government would defeat the holdout attempt using single-limb aggregation with uniform offer u^*—so that the fund would ultimately make a loss due to the fixed entry cost. Hence in the restructuring, the government ends up selecting the aggregation method according to the relative size of the bonds and the cross-bond heterogeneity in the reservation values of non-vulture investors, as illustrated in Section 3. Unless a relatively small bond is held by highly demanding creditors, the government would thus be using two-limb aggregation in equilibrium. However, if the option to use single-limb aggregation were to be removed, the

[28] When unable to restructure a bond via the triggering of CACs, governments typically end up paying large settlement amounts to the holdouts. Such settlement payments are captured by h_i in our setup.

equilibrium outcome would be dramatically different: the vulture fund would optimally decide to acquire a blocking position in one bond, making its restructuring impossible or more costly.

5 Endogenous Sorting

This section provides an explicit treatment of the heterogeneity within and across bonds, going beyond the reduced-form formulation of Section 2. We embed the government's restructuring problem in a continuous-time infinite horizon model in which bonds differ in their maturities, and the creditor base of each bond is determined endogenously in equilibrium.

5.1 Stationary Environment

Investor Heterogeneity. The market is populated by a continuum of risk-neutral investors who differ in their discount rates r, distributed according to the cumulative distribution function G on $\mathcal{R} = [r_{\min}, r_{\max}]$.

Bonds. There are two bonds S and L. The face value of bond $i \in \{S, L\}$ decays exponentially at rate δ_i. We shall assume that $\delta_S > \delta_L$, hence bonds S and L can be thought of as short-term and long-term bonds, respectively. The government may default on both bonds, and the arrival time of default is exponentially distributed with parameter η. While there is no default, bond i continuously pays at coupon rate c_i. Upon occurrence of a default, bondholders receive a bond-specific recovery rate w_i per unit of face value, as further specified below.[29] We assume that the government continuously issues new bonds, so that the relative face values of short- and long-term bonds remain constant over time. We denote by $\lambda_S \in (0, 1)$ and $\lambda_L = 1 - \lambda_S$ the relative face values of the two bonds. Moreover, we denote by $\Delta q \equiv q_S - q_L$ the price differential (per unit of face value) between the two bonds.

Restructuring. Upon default, the government offers recovery rates $\mathbf{w} = (w_S, w_L)$ and selects one of the contractually defined modification methods to implement the restructuring. We denote by $h_i(r)$ the reservation value of an investor with discount rate r holding bond i, the exact specification of which depends on the details of the microfoundation. In what follows, we use the functional form

$$h_i(r) = \frac{c_i}{r + \delta_i + \chi}, \qquad \chi \geq 0.$$

Intuitively, the parameter χ captures the extent to which the investor's reservation value is discounted relative to her subjective valuation $c_i/(r + \delta_i)$ of the bond's promised cashflow

[29] Note that the implied loss $1 - w_i$ is expressed as a fraction of the bonds' face value at the time of the restructuring, which corresponds to the notion of "market haircut" as defined in Cruces and Trebesch (2013).

stream. An investor accepts the government's restructuring offer w_i for bond i if and only if

$$r \geq \frac{c_i}{w_i} - (\delta_i + \chi).$$

Sorting Stage. Prior to the occurrence of a restructuring with anticipated recovery rate ω_i, the valuation of bond i (per unit of face value) by investor r is given by

$$Q_i(r, w_i) = \frac{c_i + \eta\, w_i}{r + \delta_i + \eta}.$$

Given anticipated recovery rates $\mathbf{w}$, investors choose which bond to hold based on their subjective valuations and the price differential Δq. The set of investors sorting into bond S is

$$\mathcal{R}_S(\Delta q, \mathbf{w}) = \big\{ r \in \mathcal{R} \,:\, Q_S(r, w_S) - Q_L(r, w_L) \geq \Delta q \big\},$$

while investors with discount rates in $\mathcal{R}_L(\Delta q, \mathbf{w}) = \mathcal{R} \setminus \mathcal{R}_S(\Delta q, \mathbf{w})$ hold bond L. We proceed under the assumption that investors take a unit position in either bond.

5.2 Equilibrium Definition

Given a partition $\big(\mathcal{R}_S, \mathcal{R}_L\big)$ in the sorting stage, the mass of investors who hold bond i is

$$\mu_i = \int_{r \in \mathcal{R}_i} dG(r),$$

and the endogenous CDF of reservation values for bond i is

$$F_i(w) = \frac{1}{\mu_i} \int_{r \in \mathcal{R}_i} \mathbf{1}_{\left\{ h_i(r) \leq w \right\}} dG(r). \tag{25}$$

In the restructuring stage, the government takes these distributions as given when choosing the recovery rates and modification method. We now define the notion of equilibrium in this two-stage stationary setting.

Definition 1. *Given the distribution G of discount rates on $\mathcal{R}$, bond characteristics (c_S, δ_S) and (c_L, δ_L), relative face values (λ_S, λ_L), default arrival rate η, and discount parameter χ, an equilibrium consists of*

(i) a price differential Δq^* and a partition $\big(\mathcal{R}_S, \mathcal{R}_L\big)$,

(ii) a modification method and a pair of recovery rates $\mathbf{w}^*$,

such that

1. *the government chooses the modification method and restructuring offers optimally given the implied distributions F_S and F_L given by (25);*

2. *investors optimally choose which bond to hold: $\mathcal{R}_i = \mathcal{R}_i(\Delta q^*, \mathbf{w}^*)$;*

3. *the market clears for each bond, $\mu_i = \lambda_i$.*

5.3 Example

We now discuss and illustrate one type of equilibrium that arises in this setting. Assuming that G is uniform on $\mathcal{R} = [0, 0.55]$, we set $(\delta_L, \delta_S) = (0.05, 0.25)$ for the decay rates, and $\eta = \chi = 0.4$ for the default intensity and discount parameter. As a baseline, we fix the relative size of the long-term bond at $\lambda_L = 0.37$. We fix the coupon rate of the short-term bond at $c_S = \mathbb{E}(r) + \delta_S$,[30] and we set $c_L = \mathbb{E}(r) + \delta_L + \epsilon_L$, where the extra parameter ϵ_L is set at zero in the baseline, but later takes value in $[-0.04, 0.04]$ to perform one comparative static exercise. The voting thresholds are set at $\tau_2^{\mathrm{s}} = 1/2$ and $\tau_2^{\mathrm{a}} = \tau_1 = 2/3$, providing for a clean comparison of aggregation methods.

For such parameters, one can construct a threshold-type equilibrium in which

$$\mathcal{R}_L = \big[r_{\min}, \widehat{r}\big] \qquad \text{and} \qquad \mathcal{R}_S = \big[\widehat{r}, r_{\max}\big].$$

In this equilibrium, long-term bonds are held by the more patient investors, with discount rates below the threshold level $\widehat{r}$. Market clearing requires that

$$G(\widehat{r}) = \lambda_L$$

and the price differential is pinned down as $\Delta q^* = Q_S\big(\widehat{r}, w_S^*\big) - Q_L\big(\widehat{r}, w_L^*\big)$, where w_i^* is the equilibrium offer to bond i. The investors' reservation values for bonds L and S in the restructuring stage lie in the intervals

$$\mathcal{V}_L = \big[h_L(\widehat{r}), h_L(r_{\min})\big] \quad \text{and} \quad \mathcal{V}_S = \big[h_S(r_{\max}), h_S(\widehat{r})\big].$$

Moreover, defining

$$\begin{aligned} a_L &= \frac{\lambda_L r_{\max} + (1-\lambda_L) r_{\min} + \delta_L + \chi}{\lambda_L (r_{\max} - r_{\min})}, \quad & b_L &= \frac{c_L}{\lambda_L (r_{\max} - r_{\min})}, \\ a_S &= \frac{r_{\max} + \delta_S + \chi}{\lambda_S (r_{\max} - r_{\min})}, & b_S &= \frac{c_S}{\lambda_S (r_{\max} - r_{\min})}, \end{aligned} \tag{26}$$

[30] This normalization implies that, absent default risk, the average investor would value the bond at par.

one can write $\mathcal{V}_i = [b_i/a_i, b_i/(a_i - 1)]$, and the reservation value CDFs are given by

$$F_i(w) = a_i - b_i/w, \qquad \text{for} \qquad w \in \mathcal{V}_i. \tag{27}$$

Thanks to the simple functional form taken by the CDFs, one can solve for the government's optimal restructuring offers under each aggregation method in closed form. The derivations follow the same logic as in Section 2 and are relegated to Appendix A.4.

Figure 3 illustrates the aforementioned equilibrium objects graphically. The left panel plots the bond valuation functions $Q_i(.\,, w_i^*)$ and the threshold value $\widehat{r}$ in the sorting stage. The center panel illustrates the reservation value functions $h_i(.)$, using solid lines in the regions of $\mathcal{R}$ where they are relevant, and dotted lines otherwise. The right panel illustrates the reservation value CDFs, revealing that bond L (in red) is the relatively "expensive" one in this example. The consent shares implied by the optimal two-limb and single-limb offers are denoted with circle and diamond markers, respectively. In the baseline parametrization, the auxiliary solution is not feasible (the optimal two-limb offer hence inducing a consent share for bond L equal to the series-by-series threshold) and the single-limb method dominates.

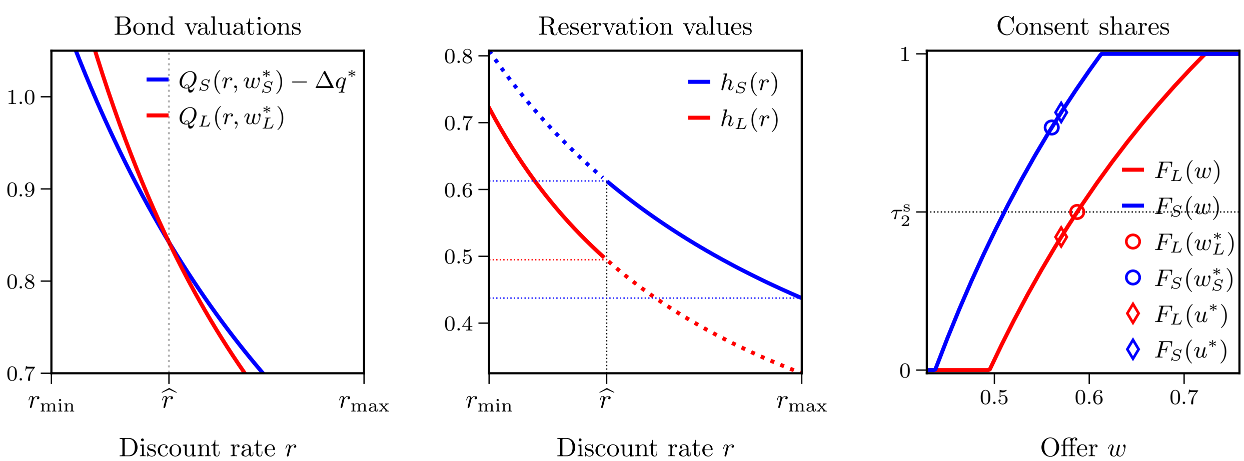

Figure 3: Equilibrium Bond Valuations and Price Differential, Reservation Values, and CDFs.

5.4 Comparative Statics

We now revisit how the optimal modification method and restructuring offers depend on the relative face value of the bonds and on the heterogeneity across reservation value distributions. In Section 3, these two aspects were independently governed by the parameters λ and γ, respectively. Now instead the degree of heterogeneity in the reservation value distributions is endogenous and partly depends on the relative bond size. Yet the insights from Section 3 are useful to interpret the comparative statics obtained in the model with sorting.

Cross-Bond Heterogeneity: the Role of Coupon Rates. We start by considering the effect of a change in the coupon rate of the long-term bond, holding everything else fixed. As captured by (26)-(27), an increase in c_L shifts F_L down and to the right, while leaving F_S unchanged. Hence, relative to the baseline case depicted in the right panel of Figure 3, the heterogeneity in equilibrium reservation value distributions increases across the two bonds.

Figure 4 illustrates the equilibrium restructuring outcomes as a function of the change ϵ_L in the coupon rate c_L relative to the baseline. The left panel plots the consent shares implied by the optimal single-limb offer (in black) and by the optimal two-limb offers (in dashed blue and dotted red for the short- and long-term bond, respectively), while the shaded colored lines represent the consent shares induced by the auxiliary solution when it is not feasible. The right panel plots the difference between the total restructuring cost under the two-limb and single-limb procedures. For low values of ϵ_L, the equilibrium reservation value distributions are quite similar, as can be seen from the fact that $F_L(u^*)$ and $F_S(u^*)$ are close to each other, and the optimal two-limb solution coincides with that of the auxiliary problem—thus dominating the single-limb solution. For larger values of ϵ_L, the auxiliary solution stops being feasible as the series-by-series constraint for bond L starts binding under two-limb aggregation, and the single-limb method becomes optimal. Hence, as in Sections 2 and 3, single-limb becomes more attractive when the reservation value distributions are (locally) more heterogeneous.

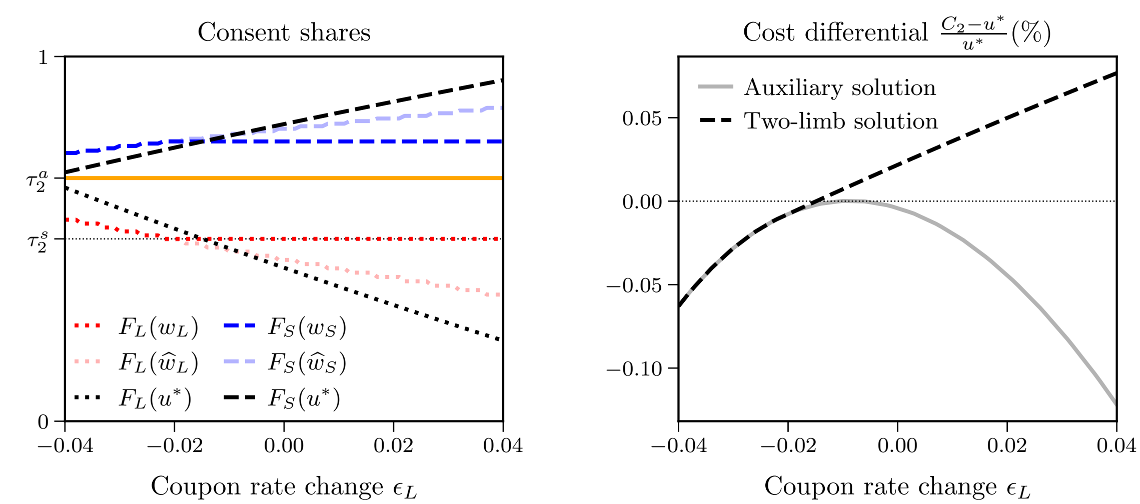

Figure 4: Comparative Static with Respect to Heterogeneity Driven by a Change in c_L.

Cross-Bond Heterogeneity: the Role of Relative Size. Next, we consider the effect of a change in the relative size λ_L of the long-term bond. Such a change affects the equilibrium outcome through two distinct channels: the pure size effect already discussed in Section 2, and a new sorting channel driven by the endogenous change in reservation value distributions.

The first channel through which a change in relative bond size affects the restructuring outcome is the one illustrated in Figure 1: when the share (λ_L) of the "expensive" bond increases, both auxiliary offers increase. Locally around the baseline, the offer $\widehat{w}_L$ to bond L is never feasible, as it implies $F_L(\widehat{w}_L) < \tau_2^{\mathrm{s}}$; however, as λ_L increases and $F_L(\widehat{w}_L)$ approaches τ_2^{s}, the two-limb method becomes cheaper than single limb, because the cost of the series-by-series constraints becomes smaller than that of the uniform applicability constraint. Through this channel, an increase in λ_L is therefore favorable to the use of the two-limb procedure.

The second effect is driven by the change in reservation value distributions arising from a change in the relative supply of bonds. As can be seen from (26)–(27), a higher share λ_L of the long-term bond lowers both the upper bound of $\mathcal{V}_S$ and the lower bound of $\mathcal{V}_L$, while it does not affect the other bounds: F_S becomes steeper, F_L flatter, and both distributions move leftwards. Figure 5 depicts with thin solid lines the partial effect driven by this "sorting channel", along with the total impact of λ_L on restructuring outcomes—depicted with thick dotted lines. The left panel plots the optimal offers under single-limb and two-limb aggregation, together with the auxiliary offers (shaded); the middle panel displays the corresponding consent shares; and the right panel depicts the restructuring cost differential.[31]

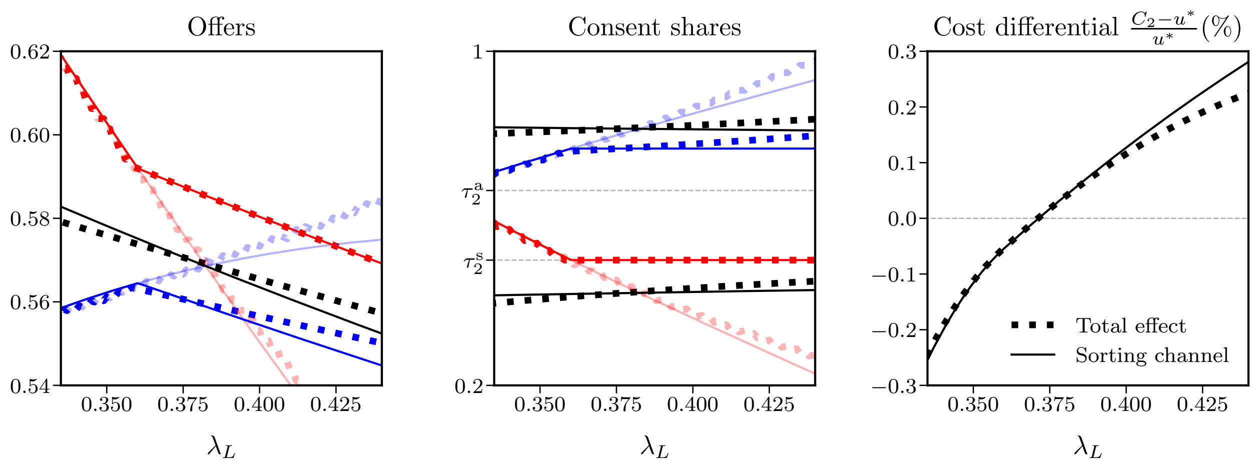

Figure 5: Comparative Static with Respect to λ_L: Sorting Channel and Total Effect.

Via the sorting channel, an increase in λ_L causes an increase in the auxiliary offer for bond S (in blue) and a decrease for bond L (in red), due to the change in the slopes of the reservation value distributions.[32] Hence the shaded solid lines representing $\widehat{w}_S$ and $\widehat{w}_L$ in the left panel have opposite slopes, and the corresponding consent shares in the center panel diverge, as in the case of an increase in heterogeneity brought about by an increase in c_L (Figure 4). For sufficiently high values of λ_L the auxiliary solution violates the series-by-series constraint for bond L, and the single-limb method becomes cheaper, as it enables the government to obtain a low consent share in the expensive bond. In this example, the pure size effect is dwarfed by the sorting channel, so the total effect closely resembles the

[31] We omit the legends to make the figure less dense; the color coding is as in Figure 4. In the middle panel, the consent shares depicted in black that are located below (resp. above) τ_2^{a} correspond to bond L (resp. S).

[32] See Section 1.2 for a characterization of the the auxiliary solution in terms of the slopes of the CDFs.

latter: a higher share of the long-term bond makes the reservation value distributions more heterogeneous, and the single-limb method more attractive.

Impact of CACs on Bond Prices. Finally, we discuss how the design of CACs can affect bond valuations, focusing on the equilibrium price differential (Δq^*) between the short- and long-term bonds. In our setting, the specification of CACs affects the price differential through its impact on the equilibrium offers in the restructuring stage.

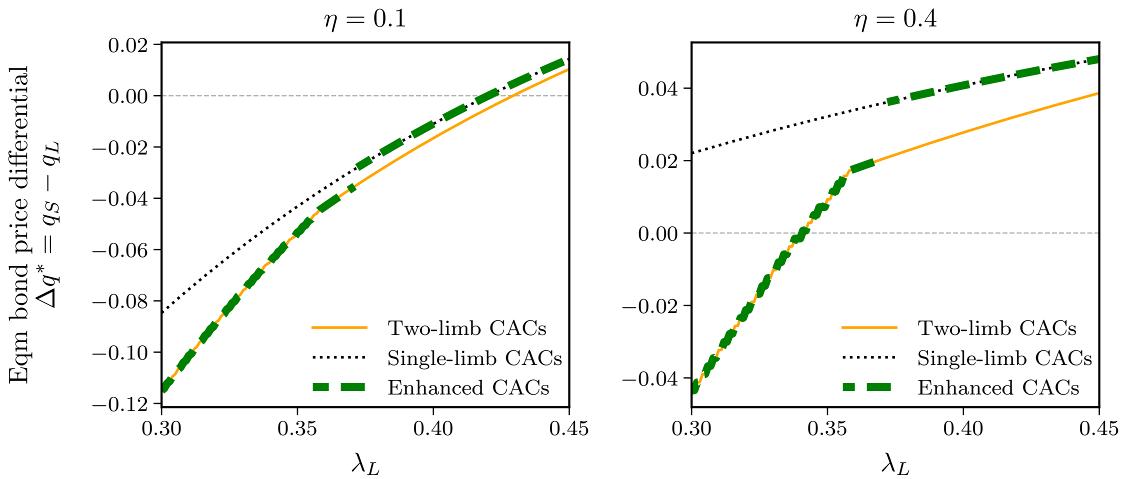

Figure 6: Bond Price Differential as Function of CACs Type, Relative Size, and Default Rate.

Figure 6 depicts Δq^* as a function of the relative size λ_L of the long-term bond under either two-limb-only CACs (solid orange), single-limb-only CACs (dotted black), or enhanced CACs (thick dashed green). The two panels are obtained for different values of the likelihood of default, η. Under any type of CACs, the relative price of the short-term bond is increasing in the relative size of the long-term bond and in the arrival rate of default. Moreover, in this example, the price differential Δq^* under single-limb CACs is always higher than under two-limb CACs. Hence, the bond price differential under enhanced CACs has a discontinuous jump upwards at the cutoff value of λ_L above which single-limb aggregation dominates. Naturally, as can be seen by comparing the two panels, the design of CACs has a greater price impact when the probability of a future restructuring is larger.

6 Conclusion

CACs constitute a key pillar of the sovereign debt architecture and determine the fate of trillions of public debt securities worldwide, should they be subject to a restructuring. This paper analyzes the optimal use of enhanced CACs by a sovereign seeking to restructure multiple bond series at minimal cost. Our analysis clarifies how the optimal restructuring approach depends on the heterogeneity across bonds and their relative size, as well as on the various voting thresholds. In particular, our analysis reveals that resorting to the single-limb procedure introduced in the last-generation ICMA CACs is optimal only under quite special circumstances, namely, when one bond is held by investors who are particularly reluctant to take a haircut and this bond is relatively small in the restructuring pool.

Our analysis also clarifies how the design of CACs and the anticipation of their optimal use affects the bonds' creditor bases and the distribution of investors within and across bonds. In particular, we show that single-limb aggregation can serve as an off-equilibrium deterrent in the presence of vulture funds, even though it may not be used in equilibrium.

Although most of our analysis is developed in a two-bond setup, it could be extended to N bonds, possibly allowing for interlocking debt stocks featuring different CACs specifications. The optimal use of sub-aggregation could be analyzed in such a setting.

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A Technical Appendix

This appendix contains the proofs and derivations of all the results stated in the main text.

A.1 Proof of Proposition 1

Define the set of offers that satisfy the aggregate constraint

$$\mathcal{A}(\tau) = \Big\{ \mathbf{w} \,:\, \sum_{i \in \mathcal{B}} \lambda_i F_i(w_i) \geq \tau \Big\},$$

the set of offers that satisfy the series-by-series constraints

$$\mathcal{S}(\tau) = \Big\{ \mathbf{w} \,:\, F_i(w_i) \geq \tau \quad \text{for all } i \in \mathcal{B} \Big\},$$

and the set of offers that satisfy the uniform applicability condition

$$\mathcal{U} = \Big\{ \mathbf{w} \,:\, w_i = w_j \quad \text{for all } i, j \in \mathcal{B} \Big\}.$$

The restructuring cost associated with the optimal single-limb offer is

$$C_1 \equiv \min_{\mathbf{w} \in \mathcal{A}(\tau_1) \cap \mathcal{U}} \boldsymbol{\lambda}.\mathbf{w} =: \boldsymbol{\lambda}.\mathbf{w}^*, \tag{A.1}$$

while the restructuring cost associated with the optimal two-limb offer is

$$C_2 \equiv \min_{\mathbf{w} \in \mathcal{A}(\tau_2^{\mathrm{a}}) \cap \mathcal{S}(\tau_2^{\mathrm{s}})} \boldsymbol{\lambda}.\mathbf{w} =: \boldsymbol{\lambda}.\mathbf{w}^{**}, \tag{A.2}$$

and the auxiliary problem can be formulated as

$$\widehat{C} \equiv \min_{\mathbf{w} \in \mathcal{A}(\tau_2^{\mathrm{a}})} \boldsymbol{\lambda}.\mathbf{w} =: \boldsymbol{\lambda}.\widehat{\mathbf{w}}.$$

Noting that $\mathcal{A}(\tau_1) \subseteq \mathcal{A}(\tau_2^{\mathrm{a}})$ since $\tau_2^{\mathrm{a}} \leq \tau_1$, condition (i) follows from the fact that

$$\mathbf{w}^* \in \mathcal{S}(\tau_2^{\mathrm{s}}) \quad \Rightarrow \quad C_1 = \min_{\mathbf{w} \in \mathcal{A}(\tau_1) \cap \mathcal{U} \cap \mathcal{S}(\tau_2^{\mathrm{s}})} \boldsymbol{\lambda}.\mathbf{w} \geq C_2, \tag{A.3}$$

while condition (ii) follows from the observation that

$$\widehat{\mathbf{w}} \in \mathcal{S}(\tau_2^{\mathrm{s}}) \quad \Rightarrow \quad \mathbf{w}^{**} = \widehat{\mathbf{w}} \quad \text{and} \quad C_2 = \widehat{C} \leq C_1. \tag{A.4}$$

Furthermore, (A.3) and (A.4) involve a strict inequality $C_2 < C_1$ if $\mathbf{w}^{**} \notin \mathcal{U}$. ∎

A.2 Proofs and Derivations for Section 2

Proof of Lemma 1. The properties of $u^* = u(\lambda, \tau_1)$ stated in the lemma follow immediately from the implicit definition (7), using the stochastic ordering assumption (6). ∎

Proof of Lemma 2. Using Lemma 1, it is immediate to see that if $F_H^{-1}(\tau_2^{\mathrm{s}}) \leq F_L^{-1}(\tau_1)$, then $u^* \geq F_H^{-1}(\tau_2^{\mathrm{s}})$ for any value of $\lambda \in (0,1)$. If instead $F_L^{-1}(\tau_1) < F_H^{-1}(\tau_2^{\mathrm{s}})$, there exists a unique $\lambda_\dagger \in (0,1)$ such that $u(\lambda_\dagger, \tau_1) = F_H^{-1}(\tau_2^{\mathrm{s}})$, and $u^* = u(\lambda, \tau_1) \geq F_H^{-1}(\tau_2^{\mathrm{s}})$ for all $\lambda \geq \lambda_\dagger$. ∎

Remark A-1. When considering two-limb aggregation in the two-bond case, the set $\mathcal{W}$ of possible offers w_L to the holders of bond L such that the aggregate consent requirement (10) can be met for some value of w_H is defined as follows

$$\mathcal{W} = \left\{ \begin{array}{ll} \mathbb{R}^+, & \text{if } \lambda \geq \tau_2^{\mathrm{a}} \\ \left(F_L^{-1}\left(\frac{\tau_2^{\mathrm{a}}-\lambda}{1-\lambda}\right), \infty \right), & \text{if } 1-\tau_2^{\mathrm{a}} \leq \lambda < \tau_2^{\mathrm{a}} \\ \left(F_L^{-1}\left(\frac{\tau_2^{\mathrm{a}}-\lambda}{1-\lambda}\right), F_L^{-1}\left(\frac{\tau_2^{\mathrm{a}}}{1-\lambda}\right) \right], & \text{if } \lambda < 1-\tau_2^{\mathrm{a}} \end{array} \right\}. \tag{A.5}$$

Remark A-2. When the auxiliary solution satisfies (15), taking into account the fact that the function g defined by (12) depends on λ, one can show that

$$\frac{d\widehat{w}_L}{d\lambda} = \left(\frac{f_L'(\widehat{w}_L)}{f_H'(\widehat{w}_H)} + \frac{1-\lambda}{\lambda} \right)^{-1} \frac{F_L(\widehat{w}_L) - \tau_2^{\mathrm{a}}}{\lambda^2 f_L(\widehat{w}_L)},$$

while

$$\frac{d\widehat{w}_H}{d\lambda} = \left(1 + \frac{1-\lambda}{\lambda} \frac{f_H'(\widehat{w}_H)}{f_L'(\widehat{w}_L)} \right)^{-1} \frac{F_L(\widehat{w}_L) - \tau_2^{\mathrm{a}}}{\lambda^2 f_L(\widehat{w}_L)}.$$

In particular, in the special case where f_H and f_L are decreasing, one can see that whenever the auxiliary solution is such that $F_L(\widehat{w}_L) > \tau_2^{\mathrm{a}}$ (or equivalently, in view of Remark 1, such that the consent share of the cheap bond is higher than for the expensive bond), a marginal increase in the relative size λ of bond H is accompanied by an improvement in the exchange offers made to both bonds.

Proof of Lemma 3. Define $J(w) = \lambda g(w) + (1-\lambda)w$. Differentiating with respect to w, one can show that the sign of $J'(w)$ coincides with the sign of $f_H\big(g(w)\big) - f_L(w)$. The second inequality in (18) is equivalent to the requirement $J'(\underline{w}) < 0$ for $\underline{w} = F_L^{-1}(\tau_2^{\mathrm{s}})$, while the second inequality in (17) is equivalent to $J'(\overline{w}) > 0$ where $\overline{w}$ is such that $g(\overline{w}) = F_H^{-1}(\tau_2^{\mathrm{s}})$. ∎

Proof of Proposition 2. This is an immediate corollary of Proposition1: indeed Lemmas 2 and 3 provide conditions such that (i) and (ii) hold, respectively. ∎

Proof of Proposition 3. We prove parts (i) and (ii) of the proposition separately.

Proof of Part (i). Throughout this part, we denote by $C_1(\lambda)$ and $C_2(\lambda)$ the restructuring cost under single-limb and two-limb voting, and we assume that the aggregate thresholds are identical under the two modification methods, namely, $\tau_1 = \tau_2^{\mathrm{a}} = \tau^{\mathrm{a}}$. It follows immediately that $\lim_{\lambda\downarrow 0} C_1(\lambda) = \lim_{\lambda\downarrow 0} C_2(\lambda) = F_L^{-1}(\tau^{\mathrm{a}})$. In the remainder of the proof, we focus on the slope of the functions C_1 and C_2 for λ close to zero. Noting that $C_1(\lambda) = u(\lambda, \tau^{\mathrm{a}})$, we apply the implicit function theorem to obtain

$$\frac{dC_1(\lambda)}{d\lambda} = \frac{\partial u(\lambda, \tau^{\mathrm{a}})}{\partial \lambda} = \frac{F_L(u^*) - F_H(u^*)}{\lambda f_H(u^*) + (1-\lambda) f_L(u^*)} > 0.$$

In particular

$$C_1'(0) = \lim_{\lambda\downarrow 0} \frac{dC_1(\lambda)}{d\lambda} = \frac{\tau^{\mathrm{a}} - F_H \circ F_L^{-1}(\tau^{\mathrm{a}})}{f_L \circ F_L^{-1}(\tau^{\mathrm{a}})}. \tag{A.6}$$

Next, we analyze the restructuring cost under two-limb voting, $C_2(\lambda)$. First, we establish that under the assumptions of Proposition 3 and for λ small, the individual consent requirement on bond H is binding. To see this, we start from the identity

$$f_L\big(F_L^{-1}(\tau^{\mathrm{a}})\big) - f_H\big(F_H^{-1}(\tau_2^{\mathrm{s}})\big) = \Big[f_L\big(F_L^{-1}(\tau^{\mathrm{a}})\big) - f_H\big(F_L^{-1}(\tau^{\mathrm{a}})\big)\Big] + \Big[f_H\big(F_L^{-1}(\tau^{\mathrm{a}})\big) - f_H\big(F_H^{-1}(\tau_2^{\mathrm{s}})\big)\Big].$$

By assumption, $F_L^{-1}(\tau^{\mathrm{a}}) < \tilde{w}$ and $f_L(w) > f_H(w)$ for all $w < \tilde{w}$, implying that the first term is strictly positive. Noting that the second term is also strictly positive under the assumptions of the proposition, we thus conclude that

$$f_L\big(F_L^{-1}(\tau^{\mathrm{a}})\big) > f_H\big(F_H^{-1}(\tau_2^{\mathrm{s}})\big). \tag{A.7}$$

In turn, the inequality (A.7) implies that (17) is violated for λ close to zero. Hence for λ close to zero, the two-limb offer is constrained by the individual consent requirement on bond H. Therefore in this neighborhood

$$C_2(\lambda) = \lambda F_H^{-1}(\tau_2^{\mathrm{s}}) + (1-\lambda) F_L^{-1}\left(\frac{\tau^{\mathrm{a}} - \lambda \tau_2^{\mathrm{s}}}{1-\lambda}\right),$$

and

$$\frac{dC_2(\lambda)}{d\lambda} = F_H^{-1}(\tau_2^{\mathrm{s}}) - F_L^{-1}\left(\frac{\tau^{\mathrm{a}} - \lambda\tau_2^{\mathrm{s}}}{1-\lambda}\right) + \frac{\tau_2^{\mathrm{a}} - \tau_2^{\mathrm{s}}}{1-\lambda}\left[f_L \circ F_L^{-1}\left(\frac{\tau^{\mathrm{a}} - \lambda\tau_2^{\mathrm{s}}}{1-\lambda}\right)\right]^{-1}.$$

In particular,

$$C_2'(0) = \lim_{\lambda \downarrow 0} \frac{dC_2(\lambda)}{d\lambda} = F_H^{-1}(\tau_2^{\mathrm{s}}) - F_L^{-1}(\tau^{\mathrm{a}}) + \frac{\tau^{\mathrm{a}} - \tau_2^{\mathrm{s}}}{f_L \circ F_L^{-1}(\tau^{\mathrm{a}})}. \tag{A.8}$$

Combining (A.6) and (A.8), we obtain that

$$C_1'(0) < C_2'(0) \quad \Leftrightarrow \quad \left(F_H^{-1}(\tau_2^{\mathrm{s}}) - F_L^{-1}(\tau^{\mathrm{a}})\right) f_L\left(F_L^{-1}(\tau^{\mathrm{a}})\right) > \tau_2^{\mathrm{s}} - F_H\left(F_L^{-1}(\tau^{\mathrm{a}})\right).$$

To show that this inequality holds, we note that f_H being strictly decreasing implies that F_H is strictly concave, which in turn implies that

$$\left(F_H^{-1}(\tau_2^{\mathrm{s}}) - F_L^{-1}(\tau^{\mathrm{a}})\right) f_H\left(F_L^{-1}(\tau^{\mathrm{a}})\right) > F_H\left(F_H^{-1}(\tau_2^{\mathrm{s}})\right) - F_H\left(F_L^{-1}(\tau^{\mathrm{a}})\right) = \tau_2^{\mathrm{s}} - F_H\left(F_L^{-1}(\tau^{\mathrm{a}})\right),$$

and the desired inequality follows from the fact that $f_L(w_L) > f_H(w_L)$ since $w_L < \tilde{w}$. By continuity, we conclude that $C_1 < C_2$ for λ close to 0.

Proof of Part (ii). Noting that $F_H^{-1}(\tau_2^{\mathrm{s}}) < F_H^{-1}(\tau_1)$, we know by assumption that the location of the crossing point $\tilde{w}$ is such that $\tilde{w} \in \left(F_L^{-1}(\tau_1), F_H^{-1}(\tau_1)\right)$. In view of Lemma 1, the intermediate value theorem implies that there exists $\tilde{\lambda} \in (0,1)$ such that $u(\tilde{\lambda}, \tau_1) = \tilde{w}$. By construction, the single-limb uniform offer u^* is equal to $\tilde{w}$ for $\lambda = \tilde{\lambda}$. Moreover, $\tilde{\lambda} F_H(\tilde{w}) + (1-\tilde{\lambda}) F_L(\tilde{w}) = \tau_1$ is equivalent to $\tilde{w} = g(\tilde{w}; \tilde{\lambda}, \tau_1)$, where $g(w; \lambda, \tau)$ denotes the unique value of w_H such that $\lambda F_H(w_H) + (1-\lambda) F_L(w) = \tau$. Therefore we can write

$$f_L(\tilde{w}) = f_H(\tilde{w}) = f_H(g(\tilde{w}; \tilde{\lambda}, \tau_1)). \tag{A.9}$$

Setting $\lambda = \tilde{\lambda}$, we next consider the restructuring cost under two-limb voting for $\tau_2^{\mathrm{a}} = \tau_1$. The assumption that the densities are decreasing guarantees convexity of the corresponding auxiliary problem, and (A.9) implies that the auxiliary solution is given by $\widehat{\mathbf{w}} = (\tilde{w}, \tilde{w})$. Yet since $F_H(\tilde{w}) < \tau_2^{\mathrm{s}}$ by assumption, the auxiliary solution violates the individual consent requirement on bond H, implying that $C_2 > \tilde{w}$. Therefore, for $\lambda = \tilde{\lambda}$ and $\tau_2^{\mathrm{a}} = \tau_1$, we have $C_1 = \tilde{w} < C_2$ and by continuity, we conclude that the single-limb method is optimal when the parameters $(\lambda, \tau_2^{\mathrm{a}})$ are close to $(\tilde{\lambda}, \tau_1)$. ∎

A.3 Derivations and Additional Results for Section 3

With a view towards applying some of the results derived in Section 2, we first note that under the parametric specification (20), the two densities are strictly decreasing, with

$$f_i(w) = \frac{1}{\phi_i} e^{-\frac{w}{\phi_i}}.$$

Moreover, the two densities cross at a single point $\tilde{w}$ given by

$$\tilde{w} = \frac{\gamma \log \gamma}{\gamma - 1} \phi_L. \tag{A.10}$$

Holding ϕ_L fixed, $\tilde{w}$ is increasing in the heterogeneity parameter γ. We also note that

$$F_L^{-1}(\tau_1) < F_H^{-1}(\tau_2^{\mathrm{s}}) \quad \Leftrightarrow \quad \gamma > \frac{\log(1-\tau_1)}{\log(1-\tau_2^{\mathrm{s}})} \equiv \overline{\gamma}_U \quad (> 1). \tag{A.11}$$

Auxiliary Solution. Under the exponential specification, condition (14) for strict convexity of the auxiliary problem is satisfied and the mapping g that captures the aggregate consent requirement becomes

$$g(w_L) = -\phi_H \log\left(\frac{1 - \tau_2^{\mathrm{a}} - (1-\lambda)e^{-w_L/\phi_L}}{\lambda}\right), \qquad w_L \in \mathcal{W}.$$

The first-order condition (15) to the auxiliary problem gives[33]

$$\widehat{w}_L = \phi_L \log\left(\frac{1+\lambda(\gamma-1)}{1-\tau_2^{\mathrm{a}}}\right), \tag{A.12}$$

and the induced consent share for bond L is given by (21). Since $F_L(\widehat{w}_H) > \tau_2^{\mathrm{a}}$, it follows from Remark 1 that $F_H(\widehat{w}_H) < \tau_2^{\mathrm{a}}$. Hence, in the context of this parametric example, only the individual constraint on bond H may ever be binding. We compute

$$\widehat{w}_H = g(\widehat{w}_L) = \phi_H \log\left(\frac{1+\lambda(\gamma-1)}{\gamma(1-\tau_2^{\mathrm{a}})}\right), \tag{A.13}$$

and the consent share for bond H is given by (22). The auxiliary offers $\widehat{w}_L$ and $\widehat{w}_H$, and the corresponding consent shares, are increasing in λ—in line with the more general result stated in Remark A-2 in Appendix A.2. Moreover, one can check that $F_L(\widehat{w}_L)$ is increasing in γ while $F_H(\widehat{w}_H)$ is decreasing in γ—that is, the spread in consent shares under the auxiliary solution is increasing in the degree of bond heterogeneity.

Optimal Two-Limb Offer. The optimal offers under two-limb voting coincide with the auxiliary solution (A.12)–(A.13) as long as $F_H(\widehat{w}_H) \geq \tau_2^{\mathrm{s}}$. Using (22), one can see that the

[33]The solution to the auxiliary problem is pinned down by this condition as long as the problem's solution is interior. In view of (A.5), one can see that the auxiliary solution is non-interior if and only if $\lambda < 1 - \tau_2^{\mathrm{a}}$ and $\gamma > (1-\lambda)/(1-\lambda-\tau_2^{\mathrm{a}})$, in which case $\widehat{w}_L = \sup \mathcal{W}$ and the consent share for bond H is zero, thus violating the individual constraint for this bond. Hence, whenever the optimal two-limb offers coincide with the auxiliary solution, the recovery values on the two bonds are given by (A.12) and (A.13).

latter inequality is equivalent to

$$\frac{1+\lambda(\gamma-1)}{\gamma} \geq \frac{1-\tau_2^{\mathrm{a}}}{1-\tau_2^{\mathrm{s}}}. \tag{A.14}$$

For given values of the voting thresholds τ_2^{a} and τ_2^{s}, one can characterize more explicitly the set of values for λ and γ such that (A.14) holds. Since $F_H(\widehat{w}_H)$ is increasing in λ and decreasing in γ, the inequality is more likely to hold for high values of λ and low values of γ. Indeed, it is easy to see that (A.14) holds

- if $\gamma \leq (1-\tau_2^{\mathrm{s}})/(1-\tau_2^{\mathrm{a}}) \equiv \overline{\gamma}_X$, for all values of λ;
- if $\lambda \geq (1-\tau_2^{\mathrm{a}})/(1-\tau_2^{\mathrm{s}}) \equiv \underline{\lambda} \in (0,1)$, for all values of γ;
- in the remainder of the parameter space for γ sufficiently small or λ sufficiently large (see Remark A-3 below for an explicit condition).

Conversely, the consent requirement on the expensive bond H is binding when there is sufficient heterogeneity across the two bonds ($\gamma > \overline{\gamma}_X$) and the relative size of bond H is small. For such parameter values, the optimal two-limb offers are

$$w_H = F_H^{-1}(\tau_2^{\mathrm{s}}) = \phi_H \log\left(\frac{1}{1-\tau_2^{\mathrm{s}}}\right), \tag{A.15}$$

$$w_L = F_L^{-1}\left(\frac{\tau_2^{\mathrm{a}}-\lambda\tau_2^{\mathrm{s}}}{1-\lambda}\right) = \phi_L \log\left(\frac{1-\lambda}{1-\tau_2^{\mathrm{a}}-\lambda(1-\tau_2^{\mathrm{s}})}\right). \tag{A.16}$$

Remark A-3. When $\gamma > \overline{\gamma}_X = \underline{\lambda}^{-1}$, the condition on λ such that (A.14) holds is

$$\lambda \geq \frac{\underline{\lambda}\gamma - 1}{\gamma - 1} \equiv \ell_X(\gamma) \in (0,1), \tag{A.17}$$

where ℓ_X is increasing in γ and converges to $\underline{\lambda}$ in the limit as γ goes to infinity. When $\lambda < \underline{\lambda}$, the condition on γ can be expressed as

$$\gamma \leq \frac{1-\lambda}{\underline{\lambda}-\lambda}, \tag{A.18}$$

where the right-hand side is increasing in λ, starting at $\overline{\gamma}_X$ for $\lambda = 0$ and going to infinity in the limit as $\lambda \uparrow \underline{\lambda}$.

Optimal Voting Procedure. When parameter values are such that (A.14) holds, two-limb aggregation is optimal since the series-by-series constraints have no bite. Yet another sufficient condition for two-limb optimality is that the optimal uniform offer u^* satisfies the series-by-series constraints, in which case the unique advantage of single-limb aggregation is worthless.

Applying Lemma 2 (see also Proposition 2), one can see that $F_H(u^*) \geq \tau_2^{\mathrm{s}}$ if $\gamma \leq \overline{\gamma}_U$, where $\overline{\gamma}_U$ is given by (A.11), or alternatively if

$$\lambda \geq \frac{1-\tau_1-(1-\tau_2^{\mathrm{s}})^\gamma}{1-\tau_2^{\mathrm{s}}-(1-\tau_2^{\mathrm{s}})^\gamma} \equiv \ell_U(\gamma) \in (0,1) \qquad \text{for } \gamma > \overline{\gamma}_U, \tag{A.19}$$

where one can check that the definition of ℓ_U ensures that $u\big(\ell_U(\gamma),\tau_1\big) = F_H^{-1}(\tau_2^{\mathrm{s}})$. It is worth noting that Condition (A.19) can also be viewed as setting an upper bound on γ that is an increasing function of λ, starting at $\overline{\gamma}_U$ for $\lambda = 0$ and going to infinity as $\lambda \uparrow (1-\tau_1)/(1-\tau_2^{\mathrm{s}})$. One can see that when $\tau_1 = \tau_2^{\mathrm{a}}$, the functions ℓ_U^{-1} and ℓ_X^{-1} both have an asymptote at $\lambda = \underline{\lambda}$, whereas if $\tau_1 > \tau_2^{\mathrm{a}}$ the asymptote for ℓ_U^{-1} is at a value of λ strictly below $\underline{\lambda}$.

In order to exploit Proposition 3 and establish instead a sufficient condition for single-limb optimality, we first need to find restrictions that ensure that $\tilde{w} \in \big(F_L^{-1}(\tau_1), F_H^{-1}(\tau_2^{\mathrm{s}})\big)$. It is immediate to see from (A.10) that this amounts to finding $\gamma > \overline{\gamma}_U$ such that

$$-\log(1-\tau_1) < \frac{\gamma\log\gamma}{\gamma-1} < -\gamma\log(1-\tau_2^{\mathrm{s}}), \tag{A.20}$$

and one can show (see Remark A-4 below) that these inequalities are satisfied for high values of γ. For such values of γ, we know from Proposition 3 that single-limb voting is optimal for λ sufficiently close to zero when $\tau_1 = \tau_2^{\mathrm{a}}$, and in the neighborhood of

$$\tilde{\lambda} = \frac{1-\tau_1-\gamma^{-\frac{\gamma}{\gamma-1}}}{(\gamma-1)\gamma^{-\frac{\gamma}{\gamma-1}}}, \tag{A.21}$$

when $\tau_1 - \tau_2^{\mathrm{a}}$ is small, where one can check that the point $\tilde{\lambda} > 0$ is such that $u(\tilde{\lambda}, \tau_1) = \tilde{w}$.

Remark A-4. To see that the inequalities in (A.20) are satisfied for high values of γ, first note that $-\log(1-\tau) > 0$ is increasing in τ, with $-\log(1-\tau) > 1$ for $\tau > 1 - e^{-1} \approx 0.63$. Since $\gamma\log\gamma/(\gamma-1)$ is strictly increasing in γ, with limit 1 as $\gamma \downarrow 1$ and going to infinity as $\gamma \to \infty$, the first inequality is satisfied for γ sufficiently large. Likewise, since $\log\gamma/(\gamma-1)$ is strictly decreasing in γ, with limit 1 as $\gamma \downarrow 1$ and going to zero as $\gamma \to \infty$, the second inequality is also satisfied for γ sufficiently large.

Necessary and Sufficient Condition. When none of the sufficient conditions holds, the participation constraint on bond H must be binding under two-limb voting, in which case the optimal two-limb offers are given by (A.15)-(A.16). The restructuring cost is then given by

$$C_2 = \lambda\phi_H \log\left(\frac{1}{\zeta_2^s}\right) + (1-\lambda)\phi_L \log\left(\frac{1-\lambda}{\zeta_2^{\mathrm{a}} - \lambda\zeta_2^{\mathrm{s}}}\right),$$

where $\zeta_2^j = 1 - \tau_2^j$ for $j \in \{\text{s}, \text{a}\}$. Under single-limb voting, the optimal uniform offer u^* is implicitly defined by

$$F(u^*) \equiv \lambda F_H(u^*) + (1-\lambda)F_L(u^*) = 1 - \lambda e^{-u^*/\phi_H} - (1-\lambda)e^{-u^*/\phi_L} = \tau_1,$$

and the total cost for the government is $C_1 = u^* = F^{-1}(\tau_1)$. Therefore, single-limb aggregation is optimal if and only if $F^{-1}(\tau_1) < C_2$, which is equivalent to $F(C_2) > \tau_1$. We thus obtain a necessary and sufficient condition for single-limb optimality in terms of parameter values:

$$\lambda\,(\zeta_2^{\text{s}})^{\lambda}\left(\frac{\zeta_2^{\text{a}} - \lambda\zeta_2^{\text{s}}}{1-\lambda}\right)^{\frac{1-\lambda}{\gamma}} + (1-\lambda)\,(\zeta_2^{\text{s}})^{\lambda\gamma}\left(\frac{\zeta_2^{\text{a}} - \lambda\zeta_2^{\text{s}}}{1-\lambda}\right)^{1-\lambda} < 1 - \tau_1. \tag{A.22}$$

Remark A-5. In the limit as $\gamma \to \infty$, Condition (A.22) boils down to

$$\lambda\,(1-\tau_2^{\text{s}})^{\lambda} < 1 - \tau_1.$$

It is easy to check that the left-hand side is strictly increasing in $\lambda \in (0,1)$ if $\tau_2^{\text{s}} \leq 1 - e^{-1} \approx 0.63$, in which case there is a unique λ_∞ such that $\lambda\,(1-\tau_2^{\text{s}})^{\lambda} = 1 - \tau_1$. In the limit as γ goes to infinity, two-limb dominates for λ greater than λ_∞.

A.4 Derivations for Section 5

Reservation Value Distributions. In the equilibrium described in Section 5.3 and under the stated parametric restrictions, market clearing requires that $\widehat{r} = r_{\min} + \lambda_L R$, where $R \equiv r_{\max} - r_{\min}$ denotes the length of $\mathcal{R}$. The reservation values for the two bonds lie in

$$\begin{aligned}
\mathcal{V}_L &= \left[\frac{c_L}{\widehat{r} + \delta_L + \chi}, \frac{c_L}{r_{\min} + \delta_L + \chi}\right] =: [\underline{w}_L, \overline{w}_L]\,, \\
\mathcal{V}_S &= \left[\frac{c_S}{r_{\max} + \delta_S + \chi}, \frac{c_S}{\widehat{r} + \delta_S + \chi}\right] =: [\underline{w}_S, \overline{w}_S]\,.
\end{aligned}$$

Using the notations $\{a_i, b_i\}_{i\in\{S,L\}}$ defined in the text by (26), one can check that $\underline{w}_i = b_i/a_i$, and $\overline{w}_i = b_i/(a_i - 1)$. For $w \in \mathcal{V}_L$, we have

$$\Pr\!\big(h_L(r) \leq w \mid r \leq \widehat{r}\big) = \frac{G(\widehat{r}) - G(c_L/w - (\delta_L + \chi))}{\lambda_L} = \frac{r_{\min} + \lambda_L R + \delta_L + \chi}{\lambda_L R} - \frac{c_L}{\lambda_L R}\frac{1}{w}.$$

Likewise, for $w \in \mathcal{V}_S$, we can write

$$\Pr\!\big(h_S(r) \leq w \mid r \geq \widehat{r}\big) = \frac{1 - G(c_S/w - (\delta_S + \chi))}{\lambda_S} = \frac{R + \delta_S + \chi + r_{\min}}{\lambda_S R} - \frac{c_S}{\lambda_S R}\frac{1}{w}.$$

Hence, using the notations $\{a_i, b_i\}_{i\in\{S,L\}}$, the CDFs for the two bonds can be written as

$$F_i(w) = \begin{cases} 0 & \text{if} \quad w < \underline{w}_i, \\ a_i - b_i/w & \text{if} \quad w \in \mathcal{V}_i, \\ 1 & \text{if} \quad w > \overline{w}_i, \qquad i \in \{S, L\}. \end{cases}$$

Auxiliary Problem and Two-Limb Offer. The auxiliary problem is

$$\min_{w_S,\, w_L} \; \lambda_S w_S + \lambda_L w_L \quad \text{subject to} \quad \lambda_S F_S(w_S) + \lambda_L F_L(w_L) = \tau_2^{\text{a}},$$

and we denote its solution by $\widehat{\mathbf{w}} = (\widehat{w}_S, \widehat{w}_L)$. In particular, when $\widehat{\mathbf{w}} \in \text{int}(\mathcal{V}_S) \times \text{int}(\mathcal{V}_L)$, the first-order optimality condition (15) implies that

$$\widehat{w}_S = \frac{\lambda_S b_S}{\lambda_S a_S + \lambda_L a_L - \lambda_L \frac{b_L}{\widehat{w}_L} - \tau_2^{\text{a}}} \qquad \text{and} \qquad \widehat{w}_L = \frac{\lambda_S b_S \sqrt{\frac{b_L}{b_S}} + \lambda_L b_L}{\lambda_S a_S + \lambda_L a_L - \tau_2^{\text{a}}}, \tag{A.23}$$

which, after substituting the expressions for $\{a_i, b_i\}_{i\in\{S,L\}}$, yields

$$\widehat{w}_S = \frac{c_S \left(\sqrt{\frac{c_L}{c_S} \frac{\lambda}{1-\lambda}} + 1 \right)}{(\lambda - \tau_2^{\text{a}})R + r_{\max} + r_{\min} + \delta_S + \delta_L + 2\chi},$$
$$\widehat{w}_L = \frac{c_L \left(\sqrt{\frac{c_S}{c_L} \frac{1-\lambda}{\lambda}} + 1 \right)}{(\lambda - \tau_2^{\text{a}})R + r_{\max} + r_{\min} + \delta_S + \delta_L + 2\chi}.$$

As in Section 2.2, the optimal two-limb offer coincides with the auxiliary solution $\widehat{\mathbf{w}}$ when the latter satisfies the series-by-series constraint for both bonds.

Single-Limb Offer. Let $F(w) = \lambda_S F_S(w) + \lambda_L F_L(w)$ denote the aggregate consent share associated with uniform offer w, with $F'(w) \geq 0$. The optimal single-limb offer is given by

$$u^* = \inf\big\{ w \geq 0 \;\big|\; F(w) = \tau_1 \big\}.$$

In particular, for parameter values such that

$$\lambda_S \in \left[\frac{t_S + \frac{c_S}{c_L}(R\tau_1 - t_L)}{R}, \frac{t_L + \frac{c_L}{c_S}(R\tau_1 - t_S)}{R} \right],$$

where $t_i = r_{\max} + \delta_i + \chi$, one can show that

$$u^* = \frac{\lambda_L b_L + \lambda_S b_S}{\lambda_L a_L + \lambda_S a_S - \tau_1} = \frac{c_L + c_S}{(\lambda - \tau_2^{\text{s}})R + r_{\max} + r_{\min} + \delta_S + \delta_L + 2\chi} \in \mathcal{V}_S \cap \mathcal{V}_L.$$

B Extension with Stochastic Consent Shares

This appendix extends the analysis of the two-bond case presented in Sections 2 and 3 to consider a situation where the government faces some uncertainty on the consent shares that a given restructuring offer may attract. We first consider a general formulation of the government's problem in the presence of uncertainty, and then provide a parametric example.

Assumptions. We proceed under the following assumptions. First, the share of consent among the holders of each bond is a random variable, whose distribution depends on the recovery rate offered to that bond. For now, we leave this distribution and its dependence on the government's offer unspecified, and allow for the possibility that consent shares may be correlated across bonds. Second, we assume that, if one or more series cannot be restructured through the activation of CACs, the bonds of these series are left unrestructured (i.e., the minimum participation thresholds is no smaller than the CAC thresholds) and the government incurs a pecuniary cost Z per unit of face value of the unrestructured series.

Under single-limb aggregation, the presence of a unique constraint implies that either both bonds are restructured, or none of them is. Under the two-limb procedure, there is the possibility that the consent shares satisfy both the aggregate constraint and the series-by-series constraint for one bond, but not that for the other bond. In this case, we assume the presence of a redesignation clause, implying that the latter bond drops out of the restructuring pool and is left unrestructured, while the former is restructured through the triggering of CACs.

Notation. We denote by $\boldsymbol{\tau} \equiv (\tau_2^{\mathrm{a}}, \tau_2^{\mathrm{s}})$ the pair of thresholds under the two-limb procedure. To formulate the problem of the sovereign in the presence of uncertainty, we denote by $p_{\mathrm{a}}(\mathbf{w}, \tau)$ the probability that a restructuring offer $\mathbf{w} = (w_H, w_L)$ attracts an aggregate consent share above the generic aggregate threshold τ, with $p_{\mathrm{a}}(u, \tau) \equiv p_{\mathrm{a}}((u, u), \tau)$; we denote by $p_{HL}(\mathbf{w}, \boldsymbol{\tau})$ the probability that the offer $\mathbf{w}$ attracts consent shares that satisfy *all* constraints under the two-limb voting rule, i.e. both the series-by-series constraints and the aggregate constraint; and we denote by $p_i(\mathbf{w}, \boldsymbol{\tau})$ the probability that *only* bond i is restructured via redesignation under two-limb, i.e. the aggregate constraint and the series-by-series constraint for bond i are satisfied, but the series-by-series constraint for bond $j \neq i$ is not satisfied. The probability that both bonds are left unrestructured under the two-limb procedure is

$$p_0(\mathbf{w}, \boldsymbol{\tau}) \equiv 1 - p_{HL}(\mathbf{w}, \boldsymbol{\tau}) - \sum_i p_i(\mathbf{w}, \boldsymbol{\tau}).$$

Government's Problem. Let $\boldsymbol{\lambda} = (\lambda_H, \lambda_L)$ denote the relative sizes of the two bonds. The cost-minimization problem of the government under two-limb aggregation is given by

$$\mathbb{E}[C_2] = \min_{\mathbf{w}} \left\{ p_{HL}(\mathbf{w}, \boldsymbol{\tau}) \, (\boldsymbol{\lambda} \cdot \mathbf{w}) + \sum_i p_i(\mathbf{w}, \boldsymbol{\tau}) \Big(\lambda_i w_i + (1 - \lambda_i) Z \Big) + p_0(\mathbf{w}, \boldsymbol{\tau}) Z \right\}.$$

Instead, under the single-limb procedure, the government's problem is given by

$$\mathbb{E}[C_1] = \min_u \Big\{p_{\mathrm{a}}(u,\tau_1)u + \big(1 - p_{\mathrm{a}}(u,\tau_1)\big)Z\Big\}.$$

Special Case. In one particular specification of the model with stochastic consent shares, an offer w_i to bond i attracts a consent share $F_i(w_i) - \nu_i$, where the noise terms (ν_H, ν_L) are distributed according to the multivariate standard normal distribution

$$\begin{bmatrix}\nu_H \\ \nu_L\end{bmatrix} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \qquad \text{where} \qquad \boldsymbol{\mu} = \begin{bmatrix}0 \\ 0\end{bmatrix}, \; \boldsymbol{\Sigma} = \begin{bmatrix}\sigma^2 & \rho\,\sigma^2 \\ \rho\,\sigma^2 & \sigma^2\end{bmatrix}.$$

It immediately follows that the expected consent shares are given by $\mathbb{E}[F_i(w_i) - \nu_i] = F_i(w_i)$. Under this specification, the aggregate consent share is $\sum_i \lambda_i (F_i(w_i) - \nu_i)$, implying that

$$p_{\mathrm{a}}(\mathbf{w},\tau) = \Phi\left(\frac{\sum_i \lambda_i F_i(w_i) - \tau}{\sigma\sqrt{\lambda_H^2 + \lambda_L^2 + 2\rho\,\lambda_H\,\lambda_L}}\right). \tag{B.1}$$

Under two limb, all consent requirements are satisfied when the following conditions hold:

$$\begin{aligned} F_i(w_i) - \tau_2^{\mathrm{s}} &\geq \nu_i, \\ F_j(w_j) - \tau_2^{\mathrm{s}} &\geq \nu_j, \\ F_i(w_i) + \frac{\lambda_j F_j(w_j) - \tau_2^{\mathrm{a}} - \lambda_j \nu_j}{\lambda_i} &\geq \nu_i, \end{aligned}$$

for $i, j \in \{H, L\},\; j \neq i$. Instead, the restructuring only includes bond i and leaves bond j unrestructured in case the second inequality is reversed. It follows that

$$\begin{aligned} p_{HL}(\mathbf{w},\boldsymbol{\tau}) &= \Pr\left(\nu_i \leq \min\left\{F_i(w_i) - \tau_2^{\mathrm{s}},\; F_i(w_i) + \frac{\lambda_j F_j(w_j) - \tau_2^{\mathrm{a}} - \lambda_j\nu_j}{\lambda_i}\right\} \;\wedge\; \nu_j \leq F_j(w_j) - \tau_2^{\mathrm{s}}\right), \\ p_i(\mathbf{w},\boldsymbol{\tau}) &= \Pr\left(\nu_i \leq \min\left\{F_i(w_i) - \tau_2^{\mathrm{s}},\; F_i(w_i) + \frac{\lambda_j F_j(w_j) - \tau_2^{\mathrm{a}} - \lambda_j\nu_j}{\lambda_i}\right\} \;\wedge\; \nu_j > F_j(w_j) - \tau_2^{\mathrm{s}}\right). \end{aligned}$$

Remark. We use the above additively separable specification to have simple expressions, even though it implies that consent shares can in principle lie outside the unit interval. This can be readily fixed by assuming that consent shares are instead given by

$$\begin{aligned} \widehat{F}_a(\mathbf{w}) &= \min\left\{\max\left\{\sum_i \lambda_i\big(F_i(w_i) - \nu_i\big),\, 0\right\},\, 1\right\}, \\ \widehat{F}_i(\mathbf{w}) &= \min\{\max\{F_i(w_i) - \nu_i,\, 0\},\, 1\}. \end{aligned}$$

This formulation delivers probabilities p_{a}, p_i, p_{HL} that are identical to those specified above, but at the cost of more involved expressions.

Illustration. We now illustrate the results of the model assuming that the expected consent share functions F_i are as per (20), with the same parameters (ϕ_L, ϕ_H, γ) as the ones set in Section 3.2. The CAC thresholds are $\tau_1 = \tau_2^{\mathrm{a}} = 2/3$ and $\tau_2^{\mathrm{s}} = 1/2$, and the cost of leaving a bond unrestructured is $Z = 7$. Figures B.1-B.3 illustrate comparative statics with respect to the relative size (λ) of the expensive bond H, the uncertainty about consent shares (σ), and their correlation (ρ), respectively. Unless specified otherwise, we set $\sigma = 0.05$ and $\rho = 0$.

In each figure, the top left panel depicts restructuring offers under two-limb voting (dashed red for bond H and dash-dotted blue for bond L) and the uniform offer under single-limb voting (dotted black). The top right panel depicts, for each aggregation method, the expected consent shares for each bond, along with the expected aggregate consent share (with the solid orange line corresponding to the expected aggregate consent under two-limb voting). The bottom right panel depicts as a dotted black line the probability $p_{\mathrm{a}}(u^*, \tau_1)$ that the restructuring goes through under single-limb, and as a solid orange line the probability $p_{HL}(\mathbf{w}, \boldsymbol{\tau})$ that the activation of CACs allows the restructuring of both bonds under two-limb voting, as well as the probabilities $p_i(\mathbf{w}, \boldsymbol{\tau})$ that only bond $i = H, L$ is being restructured. Finally, the bottom left panel depicts the expected restructuring costs $\mathbb{E}[C_2]$ and $\mathbb{E}[C_1]$ under the two-limb and single-limb procedures (solid orange and dotted black lines, respectively).

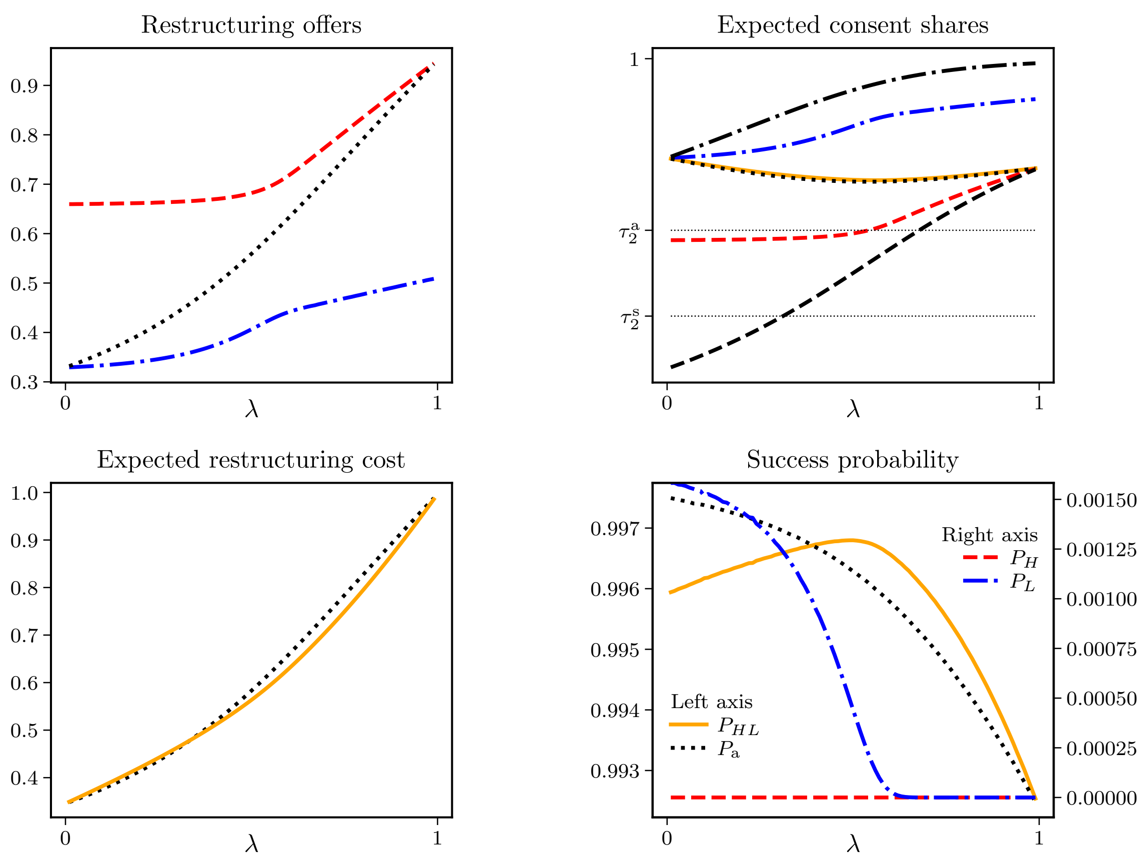

Figure B.1: Comparative Statics with Respect to λ.

Figure B.1 demonstrates that the findings illustrated in Figure 1 on the optimal restructuring approach generalize to a setting where the government faces uncertainty about consent shares. Optimal offers under either aggregation method are increasing in the share of the more demanding bond. In the presence of uncertainty, offers are made to ensure that, in expectation, consent shares are comfortably above the required thresholds, yet leaving the possibility that some bond(s) may be left unrestructured. For low values of λ, the optimal uniform offer under single limb is expected to attract from bond H a consent share below the series-by-series threshold τ_2^{s}, whereas the optimal offer to this bond under two limb is sufficiently generous to ensure that, in all likelihood, the series-by-series requirement will be met. For low values of λ, the unique advantage of single-limb voting thus makes it optimal.

Naturally, as illustrated in the top-left panel of Figure B.2, restructuring offers—as well as expected consent shares—are increasing in the degree of uncertainty. As $\sigma \downarrow 0$, offers can be tailored to just meet each of the consent requirements as an equality. When instead there is some amount of uncertainty, offers are made more generous to guarantee a wider comfort margin. Despite the optimal offers being more generous, an increase in uncertainty lowers the probability of a smooth restructuring. The bottom left panel of the figure reveals that an increase in uncertainty is favorable to the use of two-limb aggregation.

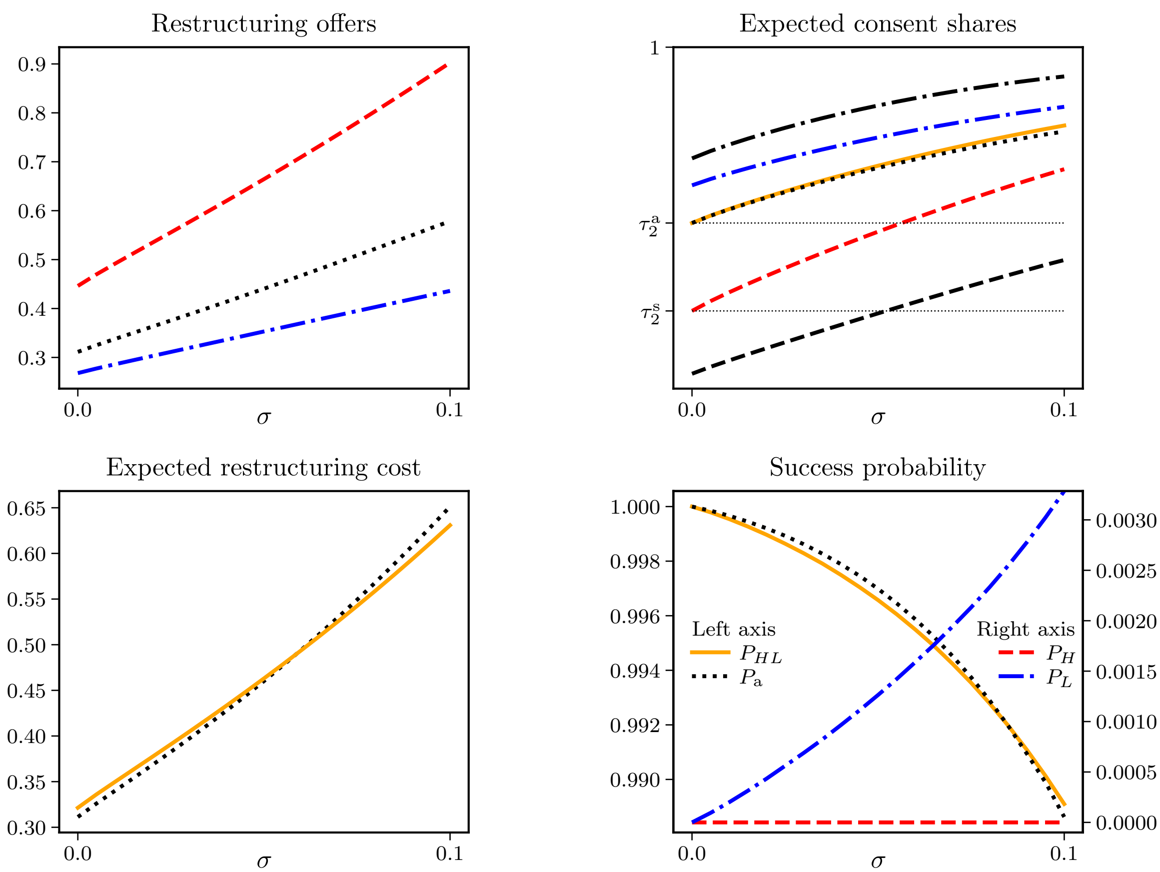

Figure B.2: Comparative Statics with Respect to σ, for $\lambda = 0.3$.

Figure B.3 illustrates the impact of the correlation of shocks to consent shares on the optimal restructuring approach. One may think of this correlation as being driven (at least partly) by the extent to which some creditors are common across the bonds. As can be seen from Equation (B.1), holding everything else constant, an increase in ρ lowers the probability of reaching a given aggregate threshold. Under single limb, despite a slight upward adjustment in the uniform offer, an increase in ρ is accompanied by an approximately linear drop in the likelihood of a smooth restructuring. Under two-limb voting, the drop is less pronounced and the probability p_{HL} of a smooth restructuring even reverts back up for high values of ρ due to the adjustment in the government's offers. Overall, an increase in the correlation of shocks to consent shares contributes to making the two-limb procedure more appealing.

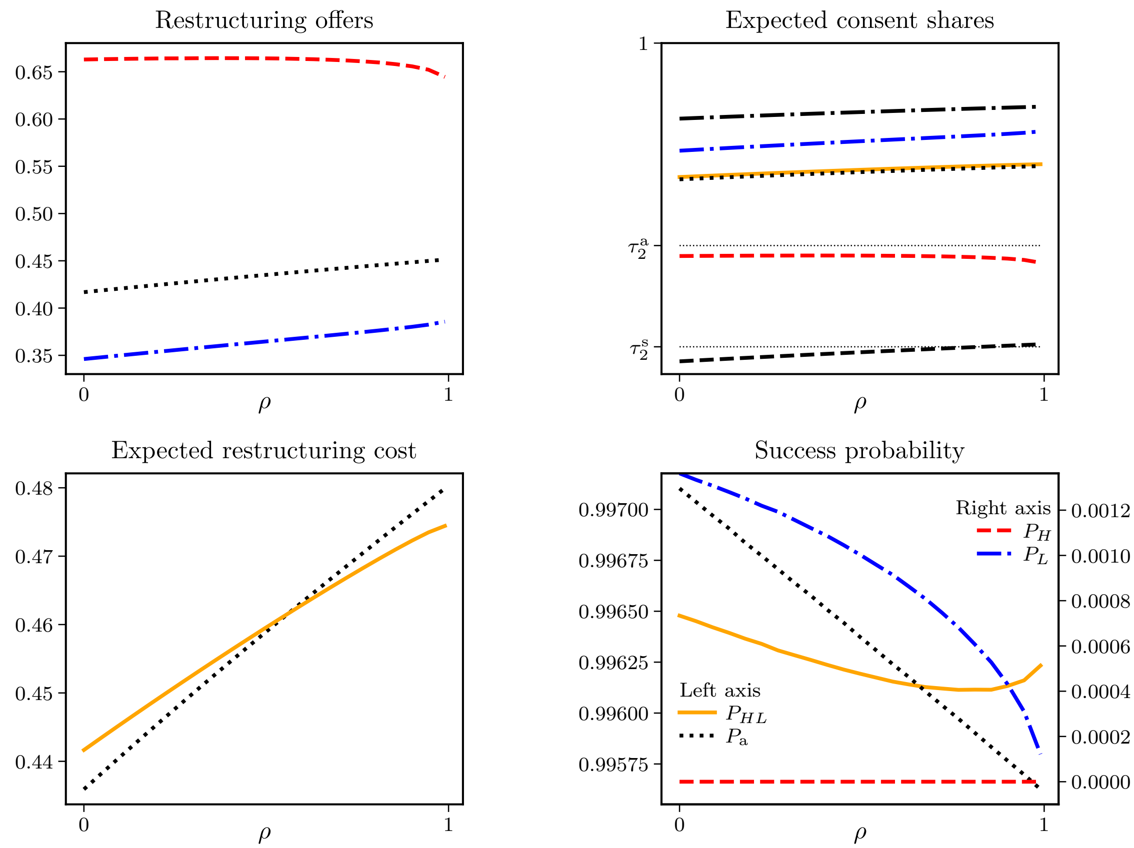

Figure B.3: Comparative Statics with Respect to ρ, for $\lambda = 0.25$.

Finally, Figure B.4 generalizes and extends the results from Sections 2 and 3 on the determinants of the optimal aggregation method. Consistent with Figure 2 (left panel, for $\tau_1 = \tau_2^{\mathrm{a}} = 2/3$), even in the presence of shocks, the single-limb procedure is found to be optimal only when there is substantial heterogeneity across bonds and the more demanding bond is relatively small. Moreover, consistent with the insights from Figures B.2 and B.3, an increase in uncertainty and in the correlation of shocks on consent shares are both favorable to the use of two-limb aggregation.

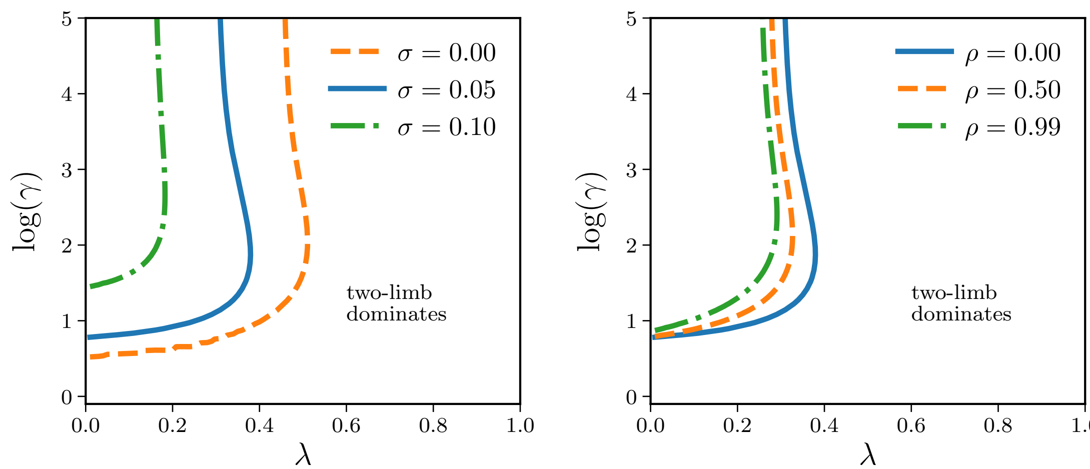

Figure B.4: Optimal Aggregation Method as a Function of Relative Size (λ) and Bond Heterogeneity (γ), for Different Levels of Volatility (σ) and Correlation (ρ) of Shocks to Consent Shares.